QUASICONFORMAL MAPPINGS OF THE UNIT DISC WITH TWO INVARIANT POINTS

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INTRODUCTION

Let Δ be the unit disc, and let $w = f(z)$ be a Q-quasiconformal mapping of Δ onto itself such that $f(0) = 0$ and $f(z_0) = z_0$ for some z_0 $(0 < |z_0| < 1)$. If $Q = 1$, then obviously $w = f(z)$ is the identity mapping. It is natural to ask how far a Q-quasiconformal mapping $w = f(z)$ satisfying the above-mentioned conditions can depart from the identity.

In this paper, we obtain a parametric representation for quasiconformal mappings of Δ onto itself that leave the points 0 and z_0 unchanged. Our results (Theorems 1 and 2) are analogues of corresponding results due to Tao-shing Shah [5]. A simple derivation of a parametric representation for quasiconformal mappings has recently been given by F. W. Gehring and E. Reich [3]. However, the variable complex dilatation as given by formula (2.1) in [3] does not imply the invariance of z_0 for changing t.

Theorems 1 and 2 enable us to obtain an estimate of $|f(z) - z|$ (Theorem 3) in terms of z, z_0, and Q for the class under consideration. In the limiting case, the estimate yields an inequality due to Tao-shing Shah [5].

1. THE CLASS $S_Q^{z_0}$ AND ITS SUBCLASSES

Let $S_Q^{z_0}$ denote the class of all functions f that map Δ onto itself Q-quasiconformally with $f(0) = 0$ and $f(z_0) = z_0$. Further, let S_* denote the class of all measurable complex dilatations μ defined a.e. in Δ and bounded by a constant less than 1. Let $(S)_*$ denote the subclass of S_* consisting of functions belonging to the class C^1 and continuable on $\overline{\Delta}$ as C^1-functions. Let $\hat{S}_*$ be the subclass of $(S)_*$ consisting of functions that have in $\overline{\Delta}$ partial derivatives of the first order subject to a global Hölder condition with a certain exponent δ $(0 < \delta \leq 1)$. Finally, let $(S)_Q^{z_0}$ and $\hat{S}_Q^{z_0}$ denote the subclasses of $S_Q^{z_0}$ consisting of functions generated by complex dilatations that belong to the classes $(S)_*$ and $\hat{S}_*$, respectively.

LEMMA 1. *The subclasses* $\hat{S}_Q^{z_0}$ *and* $(\hat{S})_Q^{z_0}$ *are dense in the class* $S_Q^{z_0}$.

The proof is analogous to the proofs in [1] and [4].

2. AN INTEGRAL LEMMA

In what follows, we consider functions f and the corresponding complex dilatations μ depending on one real parameter t.

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For an open set D, we use the notation $a(z, t) \rightrightarrows a(z)$ as $t \to 0+$ in the sense of so-called almost uniform convergence in D (that is, uniform convergence on compact subsets of D) and the convergence $\Re \frac{a(z, t)}{z} \to \Re \frac{a(z)}{z}$ on its closure.

LEMMA 2. *Suppose a complex dilatation* $\mu \in (S)_*$, *defined in* $\overline{\Delta} \times \{t: 0 < t \leq T\}$, *fulfills in* $\overline{\Delta}$ *the conditions*

$$\frac{\mu(z, t)}{t} \rightrightarrows \phi(z) \quad \textit{for } t \to 0+,$$

$$\frac{|\mu_z(z, t)|}{t} \leq k(z) \quad \textit{for } 0 < t \leq T,$$

where ϕ *and* k *are bounded. Suppose, moreover, that the function* f ($f(0, t) = 0$, $0 < t \leq T$) *generated by* μ *and mapping* $\overline{\Delta}$ *onto itself quasiconformally for* $0 < t \leq T$ *satisfies the condition* $f(z_0, t) = z_0$ ($0 < t \leq T$). *Then* $f \in (\hat{S})_Q^{z_0}$,

$$(1) \quad \frac{f(z, t) - z}{t} \rightrightarrows \frac{z(z_0 - z)}{\pi} \iint_{|\zeta| \leq 1} \left\{ \frac{\phi(\zeta)}{\zeta(z_0 - \zeta)(z - \zeta)} + \frac{\overline{\phi(\zeta)}}{\bar{\zeta}(1 - z_0\bar{\zeta})(1 - z\bar{\zeta})} \right\} d\xi\, d\eta$$

for $t \to 0+$ ($\zeta = \xi + i\eta$) *in* Δ,

and

$$(2) \quad \begin{aligned} &\frac{z_0 - z}{\pi} \iint_{|\zeta| \leq 1} \left\{ \frac{\phi(\zeta)}{\zeta(z_0 - \zeta)(z - \zeta)} + \frac{\overline{\phi(\zeta)}}{\bar{\zeta}(1 - z_0\bar{\zeta})(1 - z\bar{\zeta})} \right\} d\xi\, d\eta \\ &= \frac{1 - \bar{z}_0 z}{\pi} \iint_{|\zeta| \leq 1} \left\{ \frac{\phi(\zeta)}{\zeta(1 - \bar{z}_0\zeta)(z - \zeta)} + \frac{\overline{\phi(\zeta)}}{\bar{\zeta}(\bar{z}_0 - \bar{\zeta})(1 - z\bar{\zeta})} \right\} d\xi\, d\eta \end{aligned}$$

on $\partial\Delta$.

Proof. In the analogous lemmas of [4] and [5], it is proved first that the function β defined by

$$(3) \quad \frac{f(z, t) - z}{t} \rightrightarrows \beta(z) + \frac{1}{\pi} \iint_{|\zeta| \leq 1} \frac{\phi(\zeta)}{z - \zeta} d\xi\, d\eta$$

is holomorphic in Δ and continuous on $\overline{\Delta}$, and that

$$(4) \quad \Re \lim_{t \to 0+} \frac{f(z, t) - z}{zt} = 0 \quad \text{on } \partial\Delta.$$

The only changes in the proof arise from the replacement of the supplementary condition $f(1, t) = 1$ by $f(z_0, t) = z_0$ ($0 < t \leq T$).

Let us write $\frac{\beta(z)}{z} = \frac{b}{z} + c + h(z)$, where b and c are constants and h is holomorphic in Δ. It can easily be verified that

$$\frac{1}{2\pi i}\int_{|z'|=1}\frac{2}{z'-z}\,\Re\frac{\beta(z')}{z'}\,dz' = h(z)+\bar{b}z+2\Re c \qquad (|z|<1).$$

Hence, in view of (3) and (4),

$$\frac{\beta(z)}{z} = \frac{b}{z} - \bar{c} - \bar{b}z - \frac{1}{\pi}\iint_{|\zeta|\leq 1} z^2\,\frac{\overline{\phi(\zeta)}}{1-z\bar{\zeta}}\,d\xi\,d\eta\,. \tag{5}$$

From (3), (5), and the condition $f(0, t) = 0$ $(0 < t \leq T)$, we obtain the formula

$$b = \frac{1}{\pi}\iint_{|\zeta|\leq 1}\frac{\phi(\zeta)}{\zeta}\,d\xi\,d\eta\,,$$

and consequently

$$\lim_{t\to 0+}\frac{f(z,t)-z}{t} = -c\bar{z} + \frac{z}{\pi}\iint_{|\zeta|\leq 1}\frac{\phi(\zeta)}{\zeta(z-\zeta)}\,d\xi\,d\eta - \frac{z^2}{\pi}\iint_{|\zeta|\leq 1}\frac{\overline{\phi(\zeta)}}{\bar{\zeta}(1-z\bar{\zeta})}\,d\xi\,d\eta\,.$$

Using the condition $f(z_0, t) = z_0$ $(0 < t \leq T)$, we obtain first the equation

$$\bar{c} = \frac{1}{\pi}\iint_{|\zeta|\leq 1}\frac{\phi(\zeta)}{\zeta(z_0-\zeta)}\,d\xi\,d\eta - \frac{z_0}{\pi}\iint_{|\zeta|\leq 1}\frac{\overline{\phi(\zeta)}}{\bar{\zeta}(1-z_0\bar{\zeta})}\,d\xi\,d\eta\,,$$

and then (1). Obviously, $f \in (\hat{S})_Q^{z_0}$.

The formulae (1) and (4) imply (2), and this completes the proof.

3. PARAMETRIZATION THEOREMS

Lemma 2 implies the following theorems.

THEOREM 1. *Suppose a complex dilatation* $\mu \in \hat{S}_*$ *defined in* $\overline{\Delta}\times\{t\colon 0\leq t\leq T\}$ *has partial derivatives* μ_t *and* μ_{zt}. *Suppose, moreover, that the function* f ($f(0, t) = 0$, $0 \leq t \leq T$) *generated by* μ *and mapping* $\overline{\Delta}$ *onto itself quasiconformally for* $0 \leq t \leq T$ *satisfies the condition* $f(z_0, t) = z_0$ $(0 \leq t \leq T)$. *Then* $f \in S_Q^{z_0}$ *and*

$$\frac{\partial f}{\partial t} = \frac{f(z_0-f)}{\pi}\iint_{|\zeta|\leq 1}\left\{\frac{\phi(\zeta,t)}{\zeta(z_0-\zeta)(f-\zeta)} + \frac{\overline{\phi(\zeta,t)}}{\bar{\zeta}(1-z_0\bar{\zeta})(1-f\bar{\zeta})}\right\}d\xi\,d\eta \qquad \text{in } \Delta, \tag{6}$$

where the function ϕ *is defined by the formula*

$$\phi(\zeta,t) = \frac{u_t(f^{-1}(\zeta,t),t)}{1-|u(f^{-1}(\zeta,t),t)|^2}\exp(-2i\arg f_{\bar{\zeta}}^{-1}(\zeta,t))\,; \tag{7}$$

moreover

$$\frac{z_0 - f}{\pi} \iint_{|\zeta| \le 1} \left\{ \frac{\phi(\zeta, t)}{\zeta(z_0 - \zeta)(f - \zeta)} + \frac{\overline{\phi(\zeta, t)}}{\bar{\zeta}(1 - z_0\bar{\zeta})(1 - f\bar{\zeta})} \right\} d\xi\, d\eta$$

(8)

$$= \frac{1 - \bar{z}_0 f}{\pi} \iint_{|\zeta| \le 1} \left\{ \frac{\phi(\zeta, t)}{\zeta(1 - \bar{z}_0\zeta)(f - \zeta)} + \frac{\overline{\phi(\zeta, t)}}{\bar{\zeta}(\bar{z}_0 - \bar{\zeta})(1 - f\bar{\zeta})} \right\} d\xi\, d\eta \quad \text{on } \partial\Delta.$$

THEOREM 2. *If* $w = f(z)$ *belongs to* $\hat{S}_Q^{z_0}$, *then there exist functions* 1^0 $\omega = \phi(\zeta, t)$, *depending on* z_0 $(|z_0| \le 1)$, *defined in* $\bar{\Delta} \times \{t: 0 \le t \le T = \log Q\}$, *and having continuous partial derivatives* ϕ_z *and* $\phi_{\bar{z}}$; *and* 2^0 $\nu = \kappa(z_0, Q) > 0$, *defined for* $0 < |z_0| < 1$, *such that*

(i) $|\phi(z, t)| \le \kappa(z_0, Q)$ *in* $\bar{\Delta} \times \{t: 0 \le t \le T = \log Q\}$;

(ii) $\kappa(z_0, Q) \le 1/2$ *for* $|z_0| < 1$, *and* $\kappa(z_0, Q) \to 1/2$ *as* $|z_0| \to 1-$;

(iii) *the solution* $w = f(z, t)$ *of the equation*

$$\frac{\partial w}{\partial t} = \frac{w(z_0 - w)}{\pi} \iint_{|\zeta| \le 1} \left\{ \frac{\phi(\zeta, t)}{\zeta(z_0 - \zeta)(w - \zeta)} + \frac{\overline{\phi(\zeta, t)}}{\bar{\zeta}(1 - z_0\bar{\zeta})(1 - w\bar{\zeta})} \right\} d\xi\, d\eta \tag{9}$$

with the initial condition $f(z, 0) = z$ *is identically equal to* f.

The proofs are the same as in [5], except that the lemma applied there must be replaced by Lemma 2 of the present paper.

4. THE MAIN RESULT

In this section we obtain our estimate of $|f(z) - z|$ for the class $S_Q^{z_0}$. As usual, we let K and K' denote complete elliptic integrals; then

$$K(\sqrt{\omega}) = \frac{1}{2}\pi \left\{ 1 + \left(\frac{1}{2}\right)^2 \omega + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2 \omega^2 + \cdots \right\} \quad (|\omega| \le 1,\ \omega \ne 1),$$

and $K'(\sqrt{\omega}) = K(\sqrt{1 - \omega})$.

THEOREM 3. *If* $f \in S_Q^{z_0}$, *then*

$$|f(z) - z| \le \frac{4|z_0|}{\pi} \left| \frac{z}{z_0}\left(1 - \frac{z}{z_0}\right) \{K(\sqrt{z/z_0})K'(\sqrt{\bar{z}/\bar{z}_0}) + K(\sqrt{\bar{z}/\bar{z}_0})K'(\sqrt{z/z_0})\} \right| \times \kappa(z_0, Q) \log Q \tag{10}$$

for $|z| \le |z_0|$ *and* $z \ne z_0$, *and*

$$|f(z) - z| \le \frac{4|z|}{\pi} \left| \left(1 - \frac{z_0}{z}\right) \{K(\sqrt{z_0/z})K'(\sqrt{\bar{z}_0/\bar{z}}) + K(\sqrt{\bar{z}_0/\bar{z}})K'(\sqrt{z_0/z})\} \right| \times \kappa(z_0, Q) \log Q \tag{11}$$

for $|z_0| \le |z| \le 1$ *and* $z \ne z_0$; *the factor* κ *satisfies the inequality* $\kappa(z_0, Q) \le 1/2$ *for* $|z_0| < 1$, *and* $\kappa(z_0, Q) \to 1/2$ *as* $|z_0| \to 1-$.

Proof. In view of Lemma 1, we may assume without loss of generality that $f \in \hat{S}_Q^{z_0}$. Applying Theorem 2, we see that

$$|f^{-1}(w) - w| = \left|\int_0^T \frac{\partial f^{-1}}{\partial t}\, dt\right| \leq \int_0^T \left|\frac{\partial f^{-1}}{\partial t}\right| dt$$

$$\leq \frac{1}{\pi}\kappa(z_0, Q)\log Q \iint_{|\zeta|\leq 1} \left\{\frac{|z(z_0 - \zeta|}{|\zeta(z_0 - \zeta)(z - \zeta)|} + \frac{|z(z_0 - \zeta)|}{|\zeta(1 - \bar{z}_0\zeta)(1 - \bar{z}\zeta)|}\right\} d\xi\, d\eta,$$

where $z = f^{-1}(w)$. But

$$\iint_{|\zeta|\leq 1} \frac{d\xi\, d\eta}{|\zeta(1 - \bar{z}_0\zeta)(1 - \bar{z}\zeta)|} = \iint_{|\zeta|\geq 1} \frac{d\xi\, d\eta}{|\zeta(z_0 - \zeta)(z - \zeta)|}.$$

Thus

$$|f(z) - z| \leq \frac{|z(z_0 - z)|}{\pi}\kappa(z_0, Q)\psi(z, z_0), \tag{12}$$

where

$$\psi(z, z_0) = \iint_{-\infty}^{+\infty} \frac{d\xi\, d\eta}{|\zeta(\zeta - z_0)(\zeta - z)|}.$$

Notice now that

$$\psi(z, z_0) = \frac{\psi(z/z_0)}{|z_0|} = \frac{\psi(z_0/z)}{|z|},$$

where $\psi(\omega) = \psi(\omega, 1)$. Hence (12) implies

$$|f(z) - z| \leq \frac{|z_0|}{\pi}\left|\frac{z}{z_0}\left(1 - \frac{z}{z_0}\right)\right| \kappa(z_0, Q)\psi(z/z_0)\log Q \tag{13}$$

and

$$|f(z) - z| \leq \frac{|z|}{\pi}\left|1 - \frac{z_0}{z}\right| \kappa(z_0, Q)\psi(z_0/z)\log Q. \tag{14}$$

If $|\omega| \leq 1$ and $\omega \neq 1$, then (see [5, p. 406] and [2, p. 73])

$$\begin{aligned}\psi(\omega) &= 2\left|\Re\int_0^1 \frac{dt}{\sqrt{t(1-t)(1-\omega t)}} \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-(1-\omega)t)}}\right| \\ &= 4\left|K(\sqrt{\omega})K'(\sqrt{\bar{\omega}}) + K(\sqrt{\bar{\omega}})K'(\sqrt{\omega})\right|.\end{aligned} \tag{15}$$

From (13) and (15) we obtain immediately (10), and from (14) and (15) we obtain (11). The proof is complete.

The inequality (10) can be written in a weaker but simpler form: It is known (see Tao-shing Shah [5]) that if $f \in S_Q^1$, then

$$|f(z) - z| \leq \frac{1}{4\pi^2}\{\Gamma(1/4)\}^4 \log Q. \tag{16}$$

In the proof of this result, it is shown that

$$\max_{|\omega| \leq 1} \{\omega(\omega - 1)\psi(\omega)\} = \psi(1/2) = \frac{\{\Gamma(1/4)\}^4}{2\pi}. \tag{17}$$

The relations (10), (15), and (17) yield for $f \in S_Q^{z_0}$ the inequality

$$|f(z) - z| \leq \frac{\{\Gamma(1/4)\}^4}{2\pi^2} |z_0| \kappa(z_0, Q) \log Q \qquad (|z| \leq |z_0|,\ z \neq z_0).$$

Thus Theorem 3 implies the following corollary.

COROLLARY. *If* $f \in S_Q^{z_0}$ *and* $|z| \leq |z_0|$, $z \neq z_0$, *then*

$$|f(z) - z| \leq \frac{\{\Gamma(1/4)\}^4}{4\pi^2} |z_0| \log Q.$$

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