RIEMANNIAN MANIFOLDS OF CONSTANT NULLITY

Aaron Rosenthal

1. INTRODUCTION

Let m be a point of a (C^∞) Riemannian manifold M, and let M_m denote the tangent space to M at m. A vector $z \in M_m$ is called a *nullity vector* if $R_{xy} z = 0$ for all x, $y \in M_m$, where R_{xy} denotes the curvature transformation associated with the vectors x and y. The *nullity* $\mu(m)$ is the dimension of the space of nullity vectors at m. The purpose of this paper is to prove the following theorem concerning Riemannian manifolds with constant positive nullity.

THEOREM (*). *Let* M^n *be a complete, connected, and simply connected* C^∞ *Riemannian manifold of constant positive nullity* $\mu \leq n - 3$, *and suppose that one of the following conditions is satisfied:*

(1*) $n - \mu$ *is odd, and the sectional curvatures of all planes orthogonal to the spaces of nullity vectors are nonzero;*

(2*) *the restriction of the curvature tensor to the space of bivectors generated by vectors orthogonal to the space of nullity vectors at each* $m \in M$ *is a positive or negative definite bilinear form on this space.*

Then M^n *is a direct metric product,* $M^n = N^\mu \times C^{n-\mu}$, *where* N^μ *is complete and flat, and* $C^{n-\mu}$ *is complete.*

Nullity was defined by Chern and Kuiper [2]. Theorem (*) is a C^∞ intrinsic-manifold analogue of a theorem due to Hartman [4], who assumed the existence of an immersion.

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2. PRELIMINARIES: THE NULLITY VARIETIES AND CONULLITY OPERATORS

Throughout this paper, M will denote an n-dimensional differentiable Riemannian manifold (of class C^∞). The frame bundle, solder form, connexion form, and curvature form of $\underline{M}$ will be denoted respectively by $\overline{F}(M)$, $\overline{\theta}$, $\overline{\omega}$, and $\overline{\Omega}$ [1]. The natural projection of $\overline{F}(M)$ onto M will be denoted by $\overline{\pi}$. We shall use the index convention in which a repeated index means summation through all possible values of the index. The subspace N_m of M_m generated by the nullity vectors at m is called the *nullity space* at m. A *conullity vector* is a vector orthogonal to N_m, and the subspace C_m of conullity vectors at m is called the *conullity space* at m.

In this section we shall suppose only that the nullity is positive and constant in some open set of M. With this assumption, we can find (locally) the flat factor of the

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product structure, and when $\mu \leq n - 3$, we can establish a relation between the curvature operators of M and a set of linear operators whose vanishing will give us the second factor of the product structure.

THEOREM 2.1. *In a region* U *of* M *where the nullity is positive and constant, the distribution of nullity spaces is differentiable.*

Proof. For $m \in U$, let S_m be the linear subspace of M_m spanned by vectors of the form $R_{xy}w$, where x, y, and w belong to M_m. Then, if $z \in N_m$, the relation $0 = \langle R_{xy}z, w\rangle = \langle R_{xy}w, z\rangle$ shows that $R_{xy}w \in C_m$, so that $S_m \subset C_m$. If $S_m \neq C_m$, there is a nonzero vector $z \in C_m$ with $z \perp S_m$. But then, for all $x, y, w \in M_m$, $0 = \langle R_{xy}w, z\rangle = \langle R_{xy}z, w\rangle$, so that $z \in N_m$. Since C_m and N_m are orthogonal, it follows that $z = 0$, and that $S_m = C_m$.

For any fixed $p \in U$, let $F = (F_1, \cdots, F_n)$ be a frame field defined on a neighborhood V of p in U, and let the vector fields E_{abc} be defined by the formulas $E_{abc} = R_{F_a F_b} F_c$. The vector fields E_{abc} are differentiable in V, and using the characterization of C_m derived in the previous paragraph, we see that the vectors $E_{abc}(m)$ span C_m for each $m \in V$. At the point p, select a basis for C_p from the vectors $E_{abc}(p)$. Let us suppose that we have chosen the vectors $E_{abc}(p)$ ($(abc) \in I$, where I is an index set). Then the vector fields E_{abc} ($(abc) \in I$) are differentiable vector fields defined on V; they are independent in some (possibly smaller) neighborhood W of p, and they span C_m for each $m \in W$, because the nullity is constant in W. Since the nullity and conullity distributions are orthogonal, it follows that the nullity distribution is differentiable.

At each point m in a region U where the nullity is a positive constant, we may restrict our attention to frames that are adapted to the nullity spaces, that is, to the frames $f = (f_1, \cdots, f_\mu, f_{\mu+1}, \cdots, f_n)$ whose first μ vectors are nullity vectors and whose last $n - \mu$ vectors are conullity vectors. Hereafter, we shall use the index convention in which

$$1 \leq \alpha, \beta, \gamma, \cdots \leq \mu, \qquad \mu + 1 \leq h, i, j, \cdots \leq n, \qquad 1 \leq a, b, c, \cdots \leq n.$$

We shall refer to $(\alpha, \beta, \gamma, \cdots)$ as *nullity* indices, and to $(h, i, j, \cdots)$ as *conullity* indices.

Let F(U) be the set of adapted frames over U, and let ψ be the natural injection map of F(U) into $\overline{F}(M)$. Theorem (2.1) implies that F(U) may be given a differentiable structure such that ψ is differentiable. F(U) thus becomes a principle fiber bundle with structure group $O(\mu) \times O(n - \mu)$. When a form on $\overline{F}(M)$ is pulled back to F(U) via ψ^*, we denote the resulting form on F(U) by the symbol representing the original form, but with the bar removed. Simple calculations show that the forms θ^a are horizontal and independent on F(U). On the other hand, since the structure group of F(U) is smaller than that of $\overline{F}(M)$, some of the forms that are independent on $\overline{F}(M)$ must become dependent on the other structures when pulled back via ψ^*.

LEMMA 2.2. *The forms* ω_i^α *and* ω_α^i *are horizontal in the sense that they are dependent on the horizontal forms* θ^a.

Proof. If v is a vertical tangent to F(U), then for every α and i, $d\psi(v)$ lies in the kernel of $\bar{\omega}_i^\alpha$, which implies that v lies in the kernel of ω_i^α. Since the ω_i^α (and similarly, the ω_α^i) annihilate vertical vectors, it follows that they are horizontal.

Lemma (2.2) implies that we may write

$$\omega_i^\alpha = A_{i\beta}^\alpha \theta^\beta + B_{ij}^\alpha \theta^j, \tag{2.1}$$

$$\omega_\alpha^i = A_{\alpha\beta}^i \theta^\beta + B_{\alpha j}^i \theta^j, \tag{2.2}$$

where A and B are skew-symmetric in α and i, because $\omega_i^\alpha = -\omega_\alpha^i$.

Definition. Let f be an adapted frame at m, and let $v = v^\alpha f_\alpha$ be a nullity vector at m. Then T_v is the linear operator on C_m whose matrix with respect to the frame f is $v^\alpha B_{\alpha k}^j$. Thus

$$T_v(f_k) = v^\alpha B_{\alpha k}^j f_j \qquad (k = \mu + 1, \cdots, n).$$

We shall refer to the T_v as the *conullity operators.*

The next theorem establishes a relation between the conullity operators and the curvature transformations.

THEOREM 2.3. *Let* $m \in U$, *where* U *is a region of* M *in which the nullity is positive and constant. In addition, suppose that the nullity satisfies the inequality* $\mu \leq n - 3$ *in* U. *Then if* T *represents any one of the conullity operators* T_v $(v \in N_m)$, T *is a solution of the equation*

$$\underset{x,y,z}{\mathfrak{S}} R_{xy}(T(z)) = 0 \qquad \textit{for all } x, y, z \in C_m. \tag{2.3}$$

Proof. Let f be a frame at m, and let $z^b f_b \in N_m$. Since R_{xy} and $\overline{\Omega}$ are related by the formula $R_{xy} f_b = -\overline{\Omega}_b^a(\bar{x}, \bar{y}) f_a$, where $\bar{x}$ and $\bar{y}$ project to x and y under $d\bar{\pi}$, it follows that $z^b \overline{\Omega}_b^a = 0$. For an adapted frame f, $f_\alpha \in N_m$ for all values of α; thus $0 = \delta_\alpha^b \Omega_b^a = \Omega_\alpha^a$. That is, $\Omega_b^a = 0$ whenever the lower index is a nullity index. By skew-symmetry, $\Omega_b^a = 0$ whenever the upper index is a nullity index. Moreover, by the horizontality of Ω,

$$\Omega_j^i = R_{jk\ell}^i \theta^k \theta^\ell + 2 R_{jk\alpha}^i \theta^k \theta^\alpha + R_{j\beta\gamma}^i \theta^\beta \theta^\gamma.$$

But $f_\alpha \in N_m$, so that $0 = \delta_\alpha^b R_{bcd}^a = R_{\alpha cd}^a$, and by the symmetries of R_{bcd}^a, both $R_{jk\alpha}^i$ and $R_{j\beta\gamma}^i$ vanish. Thus Ω takes the simple form

$$\Omega_j^i = R_{jk\ell}^i \theta^k \theta^\ell. \tag{2.4}$$

When we pull back the second Bianchi identity to F(U) via ψ^*, we obtain the relation $d\Omega_b^a = \omega_b^c \Omega_c^a - \Omega_b^c \omega_c^a$. In the case where $a = i$, and $b = \beta$, this reduces to the equation $0 = \omega_\beta^j \Omega_j^i$. When previously determined values for ω_β^j and Ω_j^i are substituted from (2.2) and (2.4), this equation becomes

$$A_{\beta\gamma}^j R_{jk\ell}^i \theta^\gamma \theta^k \theta^\ell + B_{\beta h}^j R_{jk\ell}^i \theta^h \theta^k \theta^\ell = 0. \tag{2.5}$$

The 3-forms $\theta^\gamma \theta^k \theta^\ell$ and $\theta^h \theta^k \theta^\ell$ are independent, and their coefficients in (2.5) must vanish. Thus $B_{\beta h}^j R_{jk\ell}^i$ is zero when skew-symmetrized in h, k, and ℓ. But $R_{jk\ell}^i$ is already skew in k and ℓ, so that the result of the skew-symmetrization is

$$\underset{h,k,\ell}{\mathfrak{S}} B_{\beta h}^j R_{jk\ell}^i = 0. \tag{2.6}$$

But a simple calculation shows that

$$R_{f_k f_\ell}(T_{f_\beta}(f_h)) = -2 B^j_{\beta h} R^i_{jk\ell} f_i ,$$

and a comparison with (2.6) shows that any T_{f_β} satisfies the equation

$$\underset{h,k,\ell}{\mathfrak{S}} R_{f_k f_\ell}(T_{f_\beta}(f_h)) = 0 .$$

Since R and T are linear operators, f_β can be replaced by an arbitrary nullity vector, and the remaining vectors may be replaced by arbitrary elements of C_m. The result is (2.3).

COROLLARY 2.4. *Let* U *be a region of* M *where the nullity is positive and constant. Then the nullity spaces are integrable in* U, *and they generate nullity varieties that are totally geodesic and flat in their induced metric.*

Remarks. We shall not give a proof, because these results are known (Theorem 6 of [2]). We mention, however, that it suffices to show that $A^j_{\beta\gamma} = 0$ for all j, β, and γ; this follows from the fact that the coefficients of the $\theta^\gamma \theta^k \theta^\ell$ in (2.5) must vanish. Finally, we note that it will be a nullity variety that provides the flat factor for the product structure in Theorem (*).

3. THE REAL EIGENVALUES OF THE CONULLITY OPERATORS

THEOREM 3.1. *Let* m *be an arbitrary point of* M, *where* M *is a complete Riemannian manifold of constant positive nullity* μ. *Let* $v \in N_m$, *so that* T_v *is the corresponding conullity operator. Then the real eigenvalues of* T_v *vanish.*

Proof. The idea of the proof is to derive a matrix differential equation with initial value T_v. The solution of this equation will have properties that imply the result.

Since T_v is linear in v, we may suppose that v is a unit vector. Let σ be the geodesic (parametrized by arc length) starting at m with initial velocity v, and let N be the nullity variety through m. Since N is totally geodesic, σ lies in N for some neighborhood of m. Therefore σ may be extended so that it is a geodesic of N for all real t, for otherwise we could assume that $\sigma\colon (r, s) \to N$ is given as maximal. We shall show that σ can be extended as a geodesic of N to have the domain $(r, s + \delta)$, where $\delta > 0$.

Since N is totally geodesic, σ is a geodesic of M. Since M is complete, σ has an extension $\bar\sigma$ that is a geodesic of M for all real t. By the continuity of the curvature operator, $\bar\sigma'(s)$ is a nullity vector. It follows [5, Lemma 2, p. 86] that $\bar\sigma(s) \in N$. Using again the totally geodesic character of N, we see that $\bar\sigma \mid (r, s+\delta)$ is a geodesic of N, for some $\delta > 0$. Thus $\bar\sigma$ provides the desired extension of σ, and we may assume that σ is defined on the whole real line and that $\sigma'(t)$ is a nullity vector for all real t.

Let f be an adapted frame at m with $f_1 = v$, and let F be the frame field along σ obtained by parallel translation of f along σ. Then F is an *adapted* frame field on σ: the first μ vector fields of F are nullity fields because the nullity variety containing σ is totally geodesic, while the remaining vector fields of F are conullity fields because parallel translation is an isometry. Thus we may define a differentiable matrix-valued function C on the entire parameter space of σ by the formulas

$$C_j^i(t) = -B_{1j}^i(F(\sigma(t))).$$

It follows from Lemma 1 of [6] that if p is any point on σ, then there exist a neighborhood U of p and a frame field E on U such that

(1) the geodesic σ is an integral curve of E_1,

(2) the vector fields $E_1, \cdots, E_\mu$ are nullity fields,

(3) E is parallel on each integral curve of E_1, and

(4) $E = F$ on $\sigma \cap U$.

If we define the forms ϕ_b^a in U by the formulas $\phi_b^a = E^*(\omega_b^a)$, then the properties of the frame field E imply that in U

$$\nabla_{E_1} E_a = 0, \quad \nabla_{E_\alpha} E_\beta = \phi_\beta^\gamma(E_\alpha)E_\gamma \quad (\alpha \neq 1),$$

$$\nabla_{E_\alpha} E_i = \phi_i^j(E_\alpha)E_j \quad (\alpha \neq 1),$$

$$\nabla_{E_j} E_\beta = \phi_\beta^\gamma(E_j)E_\gamma + (B_{\beta j}^\gamma \circ E)E_\gamma,$$

$$\nabla_{E_j} E_i = (B_{ij}^\gamma \circ E)E_\gamma + \phi_i^k(E_j)E_k,$$

and

$$[E_1, E_j] = -\nabla_{E_j} E_1.$$

We use these equations to calculate $R_{E_1 E_j} E_i$ from the formula

$$R_{XY} Z = \nabla_{[X,Y]} Z + \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z,$$

and after a lengthy but straightforward calculation we find that

$$-R_{E_1 E_j} E_i = E_1(B_{ij}^\gamma \circ E)E_\gamma + (B_{ij}^\gamma \circ E)(B_{ik}^\gamma \circ E)E_\gamma + g^k E_k;$$

(we have not explicitly computed the coefficients g^k, because they will not enter into our discussion). Since E_1 is a nullity field, the symmetries of the curvature tensor imply that $R_{E_1 E_j} E_i = 0$, so that the coefficient of each E_a must vanish. Thus when $a = 1$, we obtain the equation

$$E_1(B_{ij}^1 \circ E) = (B_{kj}^1 \circ E)(B_{ik}^1 \circ E),$$

where we have used the skew-symmetry of B in γ and i. This equation implies that $F_1(C_j^i) = C_j^k C_k^i$, because $E = F$ on $\sigma \cap U$. But the $F_1(C_j^i)$ are the derivatives of the components of C as a function on the parameter space of σ. Thus we have the formula $(C_j^i)' = C_j^k C_k^i$, or simply $C' = C^2$. Since there is a frame field such as E in a neighborhood of each point of σ, we conclude that $C' = C^2$ on the entire real line.

A solution of this differential equation is given by the function $C(0)[I - tC(0)]^{-1}$, where I denotes the $(n - \mu) \times (n - \mu)$ identity matrix; and by a uniqueness theorem from the theory of ordinary differential equations [3, pp. 15-19], it follows that $C(t) = C(0)[I - tC(0)]^{-1}$. The form of the solution implies that the real eigenvalues of $C(0)$ vanish, for if λ is a real eigenvalue of $C(0)$, we can find a basis of M_m with the property that $C_1^1(0) = \lambda$ and $C_1^a(0) = C_a^1(0) = 0$ for all $a \neq 1$. But then $C_1^1(t) = \lambda/(1 - \lambda t)$, and C_1^1 is differentiable if and only if $\lambda = 0$. Finally, we note that $C(0)$ is the negative of the matrix of T_v relative to the frame f, and the result follows.

4. THE VANISHING OF THE CONULLITY OPERATORS

In this section we shall show that the conullity operators vanish when the conditions of Theorem (*) are satisfied. We first discuss some notation and conventions used exclusively in this section. If V is a subspace of M_m, where m is any point of M, B(V) will denote the space of bivectors generated by V, and P(V) will denote the space of planes generated by V. We say that the sectional curvature function K is *nonzero on* P(V) if the restriction of K to P(V) is nonzero. The curvature tensor gives rise to a symmetric bilinear form R on $B(M_m)$; that is, if $a = x \wedge y$ and $b = u \wedge v$ are separable bivectors, then $R(a, b) = \langle R_{xy} u, v \rangle$, and R is extended by linearity to operate on all bivectors [1, p. 162]. We also write

$$R(ijkm) = \langle R_{f_i f_j} f_k, f_m \rangle,$$

to simplify our notation.

When we say R *is definite on* B(V), we mean that the restriction of R to B(V) is a positive or negative definite bilinear form. Finally, A(B) will denote the *area of the bivector* B, and $\mathfrak{L}(v_1, \cdots, v_k)$ will denote the linear space generated by the vectors $v_1, \cdots, v_k$.

LEMMA 4.1. *Let* V *be a* 3-*dimensional subspace of* M_m, *where* m *is any point of* M, *and let* $\{v_1, v_2, v_3\}$ *be a basis for* V. *Suppose that* K *is nonzero on* P(V), *and that* b *and* c *are real numbers satisfying the relation*

$$bR(1323) = -cR(1313) + R(2323).$$

Then $b^2 + 4c > 0$.

Proof. We first note that we cannot have $b = c = 0$, for then the sectional curvature of the plane π_{23} spanned by the bivector $B_{23} = v_2 \wedge v_3$ would be zero. Thus, the bivector $B = (2cv_1 + bv_2) \wedge v_3$ is nonzero and spans a plane π. Moreover,

$$\begin{aligned} A^2(B)K(\pi) &= 4c^2 R(1313) + 4bc\,R(1323) + b^2 R(2323) \\ &= (b^2 + 4c)R(2323) \\ &= (b^2 + 4c)K(\pi_{23})A^2(B_{23}). \end{aligned}$$

But $K(\pi)$ and $K(\pi_{23})$ are nonzero and have the same sign, because K is continuous on P(V) and P(V) is a connected set. Thus $b^2 + 4c > 0$.

THEOREM 4.2. *Let* $m \in U$, *where* U *is a region of* M *in which the nullity is constant and* $0 < \mu \leq n - 3$, *and suppose that* K *is nonzero on* $P(C_m)$. *Let* T *be a* *conullity operator at* m, *and suppose that* T *has a zero eigenvalue. Then the eigenvalues of* T *are real.*

Proof. Let b and c be real numbers, and let the roots of the polynomial $p = \lambda^2 - b\lambda - c$ be two nonreal eigenvalues of T. We can choose independent conullity vectors v_1, v_2, and v_3 such that $T \mid \mathfrak{L}(v_1, v_2, v_3)$ has the matrix

$$\begin{bmatrix} 0 & c & 0 \\ 1 & b & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

in other words, such that $T(v_1) = v_2$, $T(v_2) = cv_1 + bv_2$, and $T(v_3) = 0$. Applying (2.3) to these three vectors, we find that

$$0 = R_{v_3 v_1}(cv_1 + bv_2) + R_{v_2 v_3} v_2 .$$

But the result obtained by taking the inner product of this equation with v_3 is $bR(1323) = -cR(1313) + R(2323)$, and Lemma (4.1) shows that $b^2 + 4c > 0$, which contradicts the choice of p.

LEMMA 4.3. *Let* V *be a* k*-dimensional subspace of* M_m $(k \geq 4)$, *where* m *is any point of* M, *and let* L *be a linear operator on* V *that satisfies* (2.3) *for all vectors in* V. *Suppose that* K *is nonzero on* P(V). *Then the nonreal elementary divisors of* L *are simple.*

Proof. Let b and c be real numbers, and suppose that the polynomial $\lambda^2 - b\lambda - c$ is a multiple nonreal elementary divisor of L. Then there exist independent vectors v_i (i = 1, 2, 3, 4) with the property that $L \mid \mathfrak{L}(v_1, v_2, v_3, v_4)$ has the form

$$\begin{bmatrix} 0 & c & 0 & 0 \\ 1 & b & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & b \end{bmatrix}$$

Applying (2.3) to the sets of vectors $\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$, we obtain the relations

$$0 = cR_{v_1 v_3} v_3 + bR_{v_1 v_3} v_4 + R_{v_4 v_1} v_4 + R_{v_3 v_4} v_2, \tag{4.1}$$

and

$$0 = cR_{v_2 v_3} v_3 + bR_{v_2 v_4} v_3 + R_{v_4 v_2} v_4 + cR_{v_3 v_4} v_1 + R_{v_3 v_4} v_3 . \tag{4.2}$$

(We have used the first Bianchi identity to obtain (4.2) from a more complicated equation.) Next we take the inner product of (4.1) with v_2, v_3, and v_4, and the inner product of (4.2) with v_1 and v_4. Making generous use of the symmetries of R, we obtain the set of equations

$$0 = -cR(1323) + bR(1342) + R(1424),$$

$$0 = -bR(1334) + R(1434) + R(2334),$$

$$0 = cR(1334) + R(2434),$$

$$0 = -cR(1323) + bR(1342) + R(1424) - R(1334),$$

$$0 = cR(2334) + bR(2434) + cR(1434) + R(3434).$$

From the first and fourth equations, it follows that $R(1334) = 0$. This simplifies the second and third equations, which may then be substituted into the last equation, yielding $R(3434) = 0$, contrary to the assumption that K is nonzero on P(V).

LEMMA 4.4. *Let* V *be a* k-*dimensional subspace of* M_m $(k \geq 4)$, *where* m *is any point of* M, *and let* L *be a linear operator on* V *that satisfies* (2.3) *for all vectors in* V. *Suppose that* R *is definite on* B(V), *and that every eigenvalue of* L *is pure imaginary. Then each eigenvalue of* L *is zero.*

Proof. Let c and d be real numbers with $c \leq 0$ and $d \leq 0$, and suppose that the zeros of $\lambda^2 - c$ and $\lambda^2 - d$ are pure imaginary eigenvalues of L. To show that $c = d = 0$, let us assume that $c < 0$. By Lemma (4.3), the nonreal elementary divisors of L are simple, so that we can find independent vectors v_i $(i = 1, 2, 3, 4)$ with the property that $L \mid \mathfrak{L}(v_1, v_2, v_3, v_4)$ takes the form

$$\begin{bmatrix} 0 & d & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We apply (2.3) to the set of vectors $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_3, v_4\}$, and $\{v_2, v_3, v_4\}$, and obtain four equations with which we take the following inner products: the first equation with v_3, the second with v_4, the third with v_1, and the last with v_2. The result, after we use the symmetries of R, is the set of equations

$$R(1234) = -dR(1313) + R(2323), \tag{4.3}$$

$$-cR(1234) = -dR(1414) + R(2424), \tag{4.4}$$

$$R(1234) = -cR(1313) + R(1414), \tag{4.5}$$

$$-dR(1234) = -cR(2323) + R(2424). \tag{4.6}$$

Multiplying (4.3) by $(-c)$, and (4.5) by $(-d)$, and then adding the results to (4.4) and (4.6) respectively, we find that $(c - d)R(1234) = 0$. But $R(1234) \neq 0$, because in (4.3), R(1313) and R(2323) have the same sign, while $-d \geq 0$. Thus $c = d$, and from (4.4) and (4.6) it follows that $R(1414) = R(2323)$, while from (4.3) and (4.4) it follows that $R(2424) = c^2 R(1313)$.

Next, we find the values that the curvature tensor assigns to the bivectors

$$w_1 = v_1 \wedge v_4 + v_2 \wedge v_3, \qquad w_2 = v_1 \wedge v_4 - v_2 \wedge v_3,$$

$$w_3 = cv_1 \wedge v_3 + v_2 \wedge v_4, \qquad w_4 = cv_1 \wedge v_3 - v_2 \wedge v_4.$$

After straightforward calculations we obtain the equations

$$R(w_1, w_1) = 2R(1414) + 2R(1423), \tag{4.7}$$

$$R(w_2, w_2) = 2R(1414) - 2R(1423), \tag{4.8}$$

$$R(w_3, w_3) = 2c^2 R(1313) + 2c R(1324), \tag{4.9}$$

$$R(w_4, w_4) = 2c^2 R(1313) - 2c R(1324). \tag{4.10}$$

In the positive definite case, the left-hand side of each of these equations is positive. Thus, (4.7) and (4.8) imply that

$$|R(1423)| < R(1414), \tag{4.11}$$

while (4.9) and (4.10), after division by $-c > 0$, lead to the inequality

$$|R(1324)| < -c R(1313). \tag{4.12}$$

In the negative definite case, a similar analysis yields the two relations

$$|R(1423)| < -R(1414), \tag{4.13}$$

$$|R(1324)| < c R(1313). \tag{4.14}$$

The first Bianchi identity may be written in the form

$$R(1234) = R(1324) - R(1423);$$

from this we see that

$$|R(1234)| \leq |R(1324)| + |R(1423)|. \tag{4.15}$$

In the positive definite case, substitution of (4.11) and (4.12) into (4.15) shows that

$$R(1234) \leq |R(1234)| < -c R(1313) + R(1414),$$

while in the negative definite case, substitution of (4.13) and (4.14) into (4.15) leads to the inequality

$$-R(1234) \leq |R(1234)| < c R(1313) - R(1414),$$

that is,

$$R(1234) > -c R(1313) + R(1414).$$

In either case, the resulting inequality is incompatible with (4.5). Thus $c = 0$, and the eigenvalues of L vanish.

THEOREM 4.5. *Let* $m \in U$, *where* U *is a region of* M *in which the nullity has a constant value* μ $(0 < \mu \leq n - 3)$, *and suppose that* R *is definite on* $B(C_m)$. *Then every conullity operator* T *at* m *has a real eigenvalue.*

Proof. Suppose that no eigenvalue of T is real. Since T has a real eigenvalue if C_m is odd-dimensional, we can assume that the dimension of C_m is even and not less than four. An application of Lemma (4.4) to C_m and T shows that not all of the eigenvalues of T can be pure imaginary. Thus we may assume that the zeros of the polynomials $p = \lambda^2 - 2a\lambda - d$ and $q = \lambda^2 - 2b\lambda - c$ are nonreal eigenvalues of T, where a, b, c, and d are real numbers, and where the roots of p are not pure imaginary. Since C_m and T satisfy the hypotheses of Lemma (4.3), it follows that the nonreal elementary divisors of T are simple, so that we can find independent conullity vectors v_i (i = 1, 2, 3, 4) such that $T \mid \mathfrak{L}(v_1, v_2, v_3, v_4)$ has the form

$$\begin{bmatrix} 0 & d & 0 & 0 \\ 1 & 2a & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 1 & 2b \end{bmatrix}$$

We apply (2.3) to the sets of vectors $\{v_1, v_2, v_3\}$, $\{v_1, v_3, v_4\}$, and $\{v_2, v_3, v_4\}$, and take the inner product of the resulting equations with the vectors v_3, v_2, and v_1, respectively. This gives the equations

$$R(1234) + 2a\,R(1323) = -d\,R(1313) + R(2323), \tag{4.12}$$

$$2b\,R(1324) = -c\,R(1323) + R(1424), \tag{4.13}$$

$$2a\,R(1234) + 2b\,R(1423) = -c\,R(1323) + R(1424). \tag{4.14}$$

Comparison of (4.13) and (4.14) together with the first Bianchi identity shows that $(a - b)R(1234) = 0$. If $R(1234) = 0$, then (4.12) reduces to the equation $2a\,R(1323) = -d\,R(1313) + R(2323)$. But an application of Lemma (4.1) to $\mathfrak{L}(v_1, v_2, v_3)$ leads to the inequality $a^2 + d > 0$, which contradicts the choice of p. Thus $R(1234) \neq 0$, and $a = b$.

Next, suppose that L is a linear transformation on a subspace V of M_m and that it satisfies (2.3) for all elements of V, and let r be any real scalar. Then, if I denotes the identity transformation of V, the transformation $L' = L + rI$ must also satisfy (2.3); for if x, y, and z are any vectors in V, then

$$\mathfrak{S}\{R_{xy}(L + rI)z\} = \mathfrak{S}\{R_{xy}L(z)\} + r\,\mathfrak{S}\{R_{xy}z\}.$$

But the first term on the right-hand side of this equation vanishes, by hypothesis, and the second term is zero by the first Bianchi identity.

In our particular case, $V = C_m$ and $L = T$, and we set $r = -a$, so that the matrix of T' is

$$\begin{bmatrix} -a & d & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & -a & c \\ 0 & 0 & 1 & a \end{bmatrix}$$

The zeros of the characteristic polynomial of T' are $\pm(a^2 + d)^{1/2}$ and $\pm(a^2 + c)^{1/2}$. These zeros are pure imaginary by hypothesis, and since T' must also satisfy (2.3), it follows from Lemma (4.4) that they have the value 0. Thus, in particular, $a^2 + c = 0$, which again contradicts the choice of p.

THEOREM 4.6. *Let* M *be complete, with constant positive nullity* $\mu \leq n - 3$, *and suppose that condition* (1*) *or* (2*) *of Theorem* (*) *is satisfied. Then the conullity operators of* M *vanish.*

Proof. Let T be a conullity operator. By Theorem (4.5), either condition implies that T has a real eigenvalue, which must vanish by Theorem (3.1). Theorem (4.2) implies that the remaining eigenvalues of T are real, and Theorem (3.1) implies that these eigenvalues are zero. Thus, if $T \neq 0$, there exist independent conullity vectors x, y, and z such that $T \mid \mathfrak{L}(x, y, z)$ has one of the two forms

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

An application of (2.3) to the vectors x, y, and z shows that the plane spanned by y and z has sectional curvature zero, which contradicts (1*) and (2*). Thus $T = 0$.

Remark. The vanishing of the conullity operators implies that the distribution of conullity spaces is integrable, and that the resulting integral manifolds (the *conullity varieties*) are totally geodesic. We do not need this, but it provides information on the second factor in the product structure of M that we shall obtain in the next section.

5. THE GLOBAL PRODUCT STRUCTURE OF M

In this section we shall show that the vanishing of the conullity operators allows us to apply de Rham's decomposition theorem to give M a global product structure. We follow the terminology found in [5, pp. 179-193].

Proof of Theorem (*). It suffices to show that M is reducible. However, the forms ω_α^i and ω_i^α represent the only obstruction to the reduction of the connection $\bar{\omega}$ to F(U), and the vanishing of these forms implies that the holonomy group is contained in $O(\mu) \times O(n - \mu)$.

Thus the hypotheses of de Rham's theorem are satisfied. Moreover, the parallel distributions of de Rham's theorem are precisely the nullity and conullity distributions of M, and if m is any point of M, the integral manifolds of de Rham's theorem are the nullity variety N and the conullity variety C that pass through m. It follows that M is isometric to the direct product of the nullity variety and conullity variety passing through m; that is, $M = N \times C$.

Final Remarks. (1) A simple calculation shows that the curvature of a conullity variety is given by the conullity components of the curvature of M.

(2) The validity of Theorem (*) remains an open question when $\mu = n - 2$. Entirely different techniques are needed for this case, because (2.3) no longer holds.

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University of California, Los Angeles
and
University of Colorado