REPLACEABILITY OF ℓ - ℓ METHODS OF SUMMATION

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In this note we characterize replaceability of an ℓ - ℓ method in terms of the kernel of the natural functional $\sum x_n$ on ℓ, the set of absolutely convergent series. (See [1] for terminology and notation.)

An ℓ - ℓ method is *absolutely regular* if it preserves the functional $\sigma(x) = \sum x_n$ on ℓ. An ℓ - ℓ method A is *replaceable* if there exists an absolutely regular ℓ - ℓ method B such that the corresponding absolute summability fields satisfy the inclusion relation $\ell_B \supseteq \ell_A$. Let ℓ_0 be the kernel of σ.

THEOREM. *The following statements about an* ℓ - ℓ *method are equivalent.*

(a) A *is replaceable.*

(b) *For each* $k = 1, 2, \cdots$, *the sequence* e^k *is at a positive distance from the* ℓ_A*-closure of* ℓ_0.

(c) ℓ_0 *is not* ℓ_A*-dense in* ℓ.

(d) σ *is* ℓ_A*-continuous on* ℓ.

(e) *The dual space* ℓ'_A *of* ℓ_A *contains a functional* f *such that* $f(e^k) = 1$ *for* $k = 1, 2, \cdots$.

Proof. If A is replaceable, say by B, then $B \in \ell'_A$, B vanishes on ℓ_0, and $B(e^k) = 1$ for $k = 1, 2, \cdots$; hence, each e^k lies outside the ℓ_A-closure of ℓ_0; that is, (a) ⇒ (b). (We are considering ℓ with the relative seminorm topology of ℓ_A.) If some e^k does not belong to the ℓ_A-closure of ℓ_0, then certainly ℓ_0 cannot be ℓ_A-dense in ℓ, so that obviously (b) ⇒ (c). Since ℓ_0 is the kernel of σ, it is either ℓ_A-closed or ℓ_A-dense. Hence, (c) ⇒ (d). If σ is ℓ_A-continuous on ℓ, we may extend it (by the Hahn-Banach Theorem) to some $f \in \ell'_A$ such that $f = \sigma$ on ℓ. Thus, $f(e^k) = 1$ for $k = 1, 2, \cdots$, and (d) ⇒ (e). Finally, if (e) is satisfied, then (by [1; p. 360, Lemma]) there exists an ℓ - ℓ method B such that $\ell_B \supseteq \ell_A$ and $B(x) = f(x)$ for all $x \in \ell_A$. Hence $B(e^k) = 1$ $(k = 1, 2, \cdots)$, and so A is replaceable.

REFERENCE

1. H. I. Brown and V. F. Cowling, *On consistency of* ℓ - ℓ *methods of summation*, Michigan Math. J. 12 (1965), 357-362.

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