TRANSFORMS OF CERTAIN MEASURES

Robert Kaufman

Let G be a locally compact, nondiscrete abelian group, and Γ its Pontrjagin dual. The Fourier-Stieltjes transform $\hat{\mu}$ of a measure μ is defined by the formula

$$\hat{\mu}(\gamma) = \int \overline{\gamma(x)}\,\mu(dx) \qquad (\mu \in M(G),\ \gamma \in \Gamma).$$

We present here generalizations of two theorems of Wik [2] concerning compact sets $P \subseteq G$ with the property that $\|\hat{\mu}\| \equiv \|\hat{\mu}\|_\infty = \|\mu\|$ for all $\mu \in M(P)$ (measures supported in P).

THEOREM 1. *If* $\limsup |\hat{\mu}| < \|\mu\|$ *for some* $\mu \in M(P)$, *then* $\|\hat{\lambda}\| < \|\lambda\|$ *for some* $\lambda \in M(P)$.

Here $\limsup |\hat{\mu}| = \inf_C \sup_{\gamma \notin C} |\hat{\mu}(\gamma)|$, the infimum being taken over all compact subsets C of Γ.

THEOREM 2. *Let* Γ_1 *be a closed subgroup of* Γ, *and let* Γ/Γ_1 *be compact. If*

(1) $\|\hat{\mu}\| = \|\mu\|$ *for all measures* $\mu \in M(P)$ *and*

(2) $\sup_{\gamma_1 \in \Gamma_1} |\hat{\sigma}(\gamma_1)| = \|\sigma\|$ *for all discrete measures* σ *in* P,

then

(3) $\sup_{\gamma_1 \in \Gamma_1} |\hat{\mu}(\gamma_1)| = \|\mu\|$ *for all measures in* M(P).

A general reference for the duality theory is Hewitt and Ross [1]; specific references are given below as needed. The author thanks the referee for pointing out a certain simplification in the proof of Theorem 1.

LEMMA 1. *For any measure* ν *concentrated on a countable subset* D *of* G, $\limsup |\hat{\nu}| = \|\hat{\nu}\|$.

Proof. Suppose that $\limsup |\hat{\nu}| < \|\hat{\nu}\|$; then $|\hat{\nu}(\gamma_0)| = \|\hat{\nu}\|$ for some $\gamma_0 \in \Gamma$. There exist a compact set $C \subseteq \Gamma$ and a positive number δ such that $|\hat{\nu}| < \|\hat{\nu}\| - \delta$ in the complement of C. Since ν is an atomic measure, there exist a finite set $\{d_1, d_2, \cdots, d_n\} \subseteq D$ and a positive number ε such that whenever

$$\gamma \in \Gamma \quad \text{and} \quad |\gamma(d_i) - 1| < \varepsilon \quad (1 \le i \le n),$$

then $|\hat{\nu}(\gamma + \gamma_0) - \hat{\nu}(\gamma_0)| < \delta$, whence $\gamma + \gamma_0 \in C$.

If χ is a character of G, not assumed to be continuous, then χ is in the pointwise closure of the set

$$C_1 = \{\gamma \in \Gamma\colon |\gamma(d_i) - \chi(d_i)| < \varepsilon/2,\ 1 \le i \le n\}$$

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(see [1, p. 432, Section (26.16)]). But C_1 has compact closure in the topology of Γ, so that χ is necessarily continuous. This evidently implies that every subgroup of G, and in particular the countable subgroup G_0 generated by D, is closed. Now the annihilator $G_0^\perp$ is compact, so that G_0 is open in G [1, p. 365, Section (23.25)]. This clearly conflicts with Baire's Theorem; the lemma is thereby proved.

Proof of Theorem 1. We can suppose that P is uncountable and therefore contains a countably infinite subset B such that $|\mu|(B) = 0$, with a point of accumulation b_0. Suppose also that $\limsup |\hat{\mu}| < r < \|\hat{\mu}\|$, so that $E = \{\gamma: r \leq |\hat{\mu}(\gamma)|\}$ is compact. By the definition of Pontrjagin's topology in Γ, there exists for each $n \geq 1$ a neighborhood U_n of b_0 such that $|\gamma(x) - \gamma(b_0)| < 1/n$ for all γ in E and x in U_n. Let b_n and b_n' be distinct elements of $U_n \cap B \subseteq P$, and let λ_n be the measure with mass 1 at b_n and -1 at b_n'. For every n, $\|\mu + \lambda_n\| = 2$, while

$$\|\hat{\mu}(\gamma) + \hat{\lambda}_n(\gamma)\| \leq 2 + r \quad \text{if } \gamma \notin E.$$

Finally, $\hat{\lambda}_n + \hat{\mu}$ converges to $\hat{\mu}$ uniformly on E, and

$$\limsup_{n \to \infty} \|\hat{\mu} + \hat{\lambda}_n\| \leq 2 + r < \|\mu\| + 2,$$

as required.

Proof of Theorem 2. We suppose that $0 \in P$. The hypothesis on Γ_1 is that $\Gamma_1 = Z^\perp$ for a discrete subgroup Z of G. For a neighborhood U of 0 in Γ, the cosets $\Gamma_1 + U$ form a neighborhood U^* of the identity in the compact group Γ/Γ_1. The open set U^* contains the subgroup orthogonal to a finitely generated subgroup Z_0 of Z, whence $Z_0^\perp \subseteq \Gamma_1 + U$. Inasmuch as Fourier-Stieltjes transforms are uniformly continuous on Γ, we can assume that $\Gamma_1^\perp$ is in fact finitely generated. The following consequence will be useful: for each closed subgroup H of Γ/Γ_1 there exists a neighborhood V of 0 in Γ/Γ_1 such that $H + V$ contains no subgroup larger than H. Less formally, Γ/Γ_1 and its factor groups are free of "small subgroups."

To prove Theorem 2, it is sufficient (and in fact necessary) to verify that for each $\gamma \in \Gamma$ and for each measure $\mu \geq 0$ in P there exists a sequence $\{\gamma_n\} \subseteq \Gamma_1$ such that

$$\int |\gamma_n(x) - \gamma(x)|\, \mu(dx) \to 0 .$$

Since hereafter only uniform convergence or a weaker convergence is considered, we tacitly assume that Γ is metrizable [1, p. 70, Section (8.3)].

Henceforth, σ is a fixed positive measure in P with positive mass at 0; for each measurable subset $T \subseteq P$, we define a subgroup $I(T) \subseteq \Gamma/\Gamma_1$ as follows: An element θ of Γ/Γ_1 belongs to I(T) if there exists a sequence $\{\gamma_n\} \subseteq \Gamma_1$ such that

$$\int_T |\gamma_n(x) - 1|\, \sigma(dx) \to 0 \quad \text{and} \quad \gamma_n + \Gamma_1 \to \theta .$$

Clearly, I(T) is a closed subgroup.

LEMMA 2. *There exists an* $\varepsilon_0 > 0$ *such that* $\sigma(T) > \sigma(P) - \varepsilon_0$ *implies* $I(T) = I(P)$.

Proof. Suppose $T_n \subseteq P$ $(n = 1, 2, \cdots)$, $\sigma(T_n) \to \sigma(P)$, $\theta_n \in I(T_n)$, and $\theta_n \to \theta$. For each n, there exists an element γ_n in Γ such that

$$\int_{T_n} |\gamma_n(x) - 1| \, \sigma(dx) \leq 1/n, \qquad d(\gamma_n + \Gamma_1, \theta_n) \leq 1/n.$$

Therefore $\gamma_n + \Gamma_1 \to \theta$, while

$$\int_P |\gamma_n(x) - 1| \, \sigma(dx) \leq 2\sigma(T_n') + \int_{T_n} |\gamma_n(x) - 1| \, \sigma(dx) \to 0.$$

Thus $\theta \in I(P)$, that is, $\lim\sup \{I(T_n): \sigma(T_n) \to \sigma(P)\} = I(P)$. The lemma is now a consequence of the property of Γ/Γ_1 mentioned at the beginning of the proof.

To complete the proof of Theorem 2, let D be the countable set on which the discrete part of σ is concentrated. For each character $\gamma \in \Gamma$, there exists by hypothesis a sequence $\{\gamma_n\} \subseteq \Gamma_1$ such that

$$\left| \int_D \overline{\gamma_n(x)} \gamma(x) \, \sigma(dx) \right| \to \int_D \sigma(dx).$$

Since σ has positive measure at 0, we conclude that $\int_D |\gamma_n(x) - \gamma(x)| \, \sigma(dx) \to 0$.

Among the measurable subsets T $(P \supseteq T \supseteq D)$ there exists a subset T_0 that has the approximation property just described for D and is contained in no subset of larger σ-measure having this property. (This can be proved by an argument similar to that used in Lemma 1.) Theorem 2 is simply the assertion that $\sigma(T_0) = \sigma(P)$; let us assume the contrary and obtain a contradiction. Since $T_0 \supseteq D$, σ is a continuous measure in T_0' (the prime indicates the complement); therefore $0 < \sigma(T_1) < \varepsilon_0$, for some subset T_1 of T_0'.

Let $\beta \in \Gamma$, and define a measure

$$\nu(E) \equiv \int_{T_1 \cap E} \beta(x) \, \sigma(dx) + \sigma(E \cap T_1').$$

There exists a sequence $\{\gamma_n\} \subseteq \Gamma$ such that $|\hat{\nu}(\gamma_n)| \to \|\nu\| = \sigma(P)$. From the condition $\sigma\{0\} > 0$ we conclude that

$$\int_{T_1'} |\gamma_n(x) - 1| \, \sigma(dx) \to 0 \quad \text{and} \quad \int_{T_1} |\gamma_n(x) - \beta(x)| \, \sigma(dx) \to 0.$$

We may suppose that $\gamma_n + \Gamma_1 \to \theta$, and we use the fact that $\sigma(T_1') > \sigma(P) - \varepsilon_0$, whence $\theta \in I(P) = I(T_1')$, by Lemma 2. For some sequence $\{\gamma_n'\} \subseteq \Gamma$, $\int_P |\gamma_n'(x) - 1| \, \sigma(dx) \to 0$ while $\gamma_n' + \Gamma_1 \to \theta$. Using the metrizability of Γ, we choose a third sequence $\{\gamma_n^*\} \subseteq \Gamma_1$ for which $(\gamma_n' - \gamma_n) + \gamma_n^* \to 0$. Then

$$\int_{T_1'} |\gamma_n^*(x) - 1| \, \sigma(dx) \to 0, \qquad \int_{T_1} |\gamma_n^*(x) - \beta(x)| \, \sigma(dx) \to 0.$$

Finally, let $\alpha \in \Gamma$. There exists a sequence $\{\chi_n\} \subseteq \Gamma_1$ such that

$$\int_{T_0} |\alpha(x) - \chi_n(x)|\, \sigma(dx) \leq 1/n \qquad (n = 1, 2, \cdots)\,.$$

In the previous calculation, we set $\beta = \bar{\chi}_n \alpha$, and we thus obtain characters $\chi_n' \subseteq \Gamma_1$ such that

$$\int_{T_0} |\chi_n'(x) - 1|\, \sigma(dx) \leq 1/n \quad \text{and} \quad \int_{T_1} |\chi_n'(x) - \overline{\chi_n(x)}\alpha(x)|\, \sigma(dx) \leq 1/n\,.$$

Then

$$\int_{T_i} |\alpha(x) - \chi_n(x)\chi_n'(x)|\, \sigma(dx) \to 0 \qquad (i = 0, 1)\,.$$

Consequently, $T_0 \cup T_1$ has the approximation property; this contradiction completes the proof of Theorem 2.

REFERENCES

1. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. I: *Structure of topological groups, integration theory, group representations*, Die Grundlehren der mathematischen Wissenschaften, Bd. 115, Academic Press, New York, 1963.
2. I. Wik, *On linear dependence in closed sets*, Ark. Math. 4 (1963), 209-218 (1961).

University of Illinois
Urbana, Illinois