PIECEWISE LINEAR UNKNOTTING OF $S^p \times S^q$ IN S^{p+q+1}

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INTRODUCTION

Denote by S^n the unit n-sphere in euclidean $(n+1)$-space, and by D^n the unit n-ball in euclidean n-space.

J. W. Alexander [2] proved that if $S^1 \times S^1$ is piecewise linearly embedded in S^3, then the closure of one of the components of $S^3 - S^1 \times S^1$ is homeomorphic to $S^1 \times D^2$. Alexander's method is based on the study of the intersections of a plane with $S^1 \times S^1$ as the plane moves through euclidean 3-space.

A. Kosinski [6] generalized this result by showing that every product $S^p \times S^q$ differentiably embedded in S^{p+q+1} can be unknotted differentiably in S^{p+q+1}, provided $p > q > 1$, $p + q > 5$, and p is odd in case $q = 2$. For this, he used Smale theory to show that one of the components of $S^{p+q+1} - S^p \times S^q$ is diffeomorphic to $S^q \times D^{p+1}$. Then, using the fact that S^q unknots differentiably in S^{p+q+1} under the above assumptions on p and q, he was able to unknot $S^p \times S^q$ in S^{p+q+1}. He asked whether the same result is true in the PL (piecewise linear) category. Our purpose is to answer this question and to drop some of the condition on p and q. (This result has been proved independently by C. T. C. Wall.)

Reformulating Alexander's theorem, we can say that if $S^1 \times S^1$ is PL embedded in S^3, then the closure of one of the components is a regular neighborhood of some 1-sphere embedded in S^3. We generalize this reformulation, by proving that if there is a locally unknotted PL embedding of $S^p \times S^q$ in S^{p+q+1}, where $p \geq q > 1$ and $p + q > 4$, then the closure of one of the components of $S^{p+q+1} - S^p \times S^q$ is a regular neighborhood of a p-sphere embedded in S^{p+q+1}. Using Zeeman's unknotting theorem and Whitehead's regular neighborhood theorem [12], we can then show that $S^p \times S^q$ unknots in S^{p+q+1}. We need the restriction that the embedding be locally unknotted, since the Schoenflies conjecture has not been proved in the piecewise linear category for dimension greater than 3; thus there is an essential difficulty in local unknotting.

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1. THE PIECEWISE LINEAR CATEGORY

Throughout this paper, all *simplicial complexes* shall be finite simplicial complexes. Sometimes we shall revert to polyhedra, in order to avoid excessive subdivision. By a *polyhedron* we mean the space underlying a finite simplicial complex; and by a *subpolyhedron of a simplicial complex*, we mean the subspace underlying a subcomplex of some rectilinear subdivision.

1.1. *Definition.* If K and L are simplicial complexes, then a map $f: K \to L$ is said to be *piecewise linear* if there exist rectilinear subdivisions K' and L' of K

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and L such that f is a simplicial map from K' to L'. If, in addition, such a map f is a homeomorphism onto, then K and L are said to be *piecewise linearly homeomorphic.*

1.2. *Definition.* An n-*ball* is any simplicial complex piecewise linearly homeomorphic to an n-simplex. An n-*sphere* is any simplicial complex piecewise linearly homeomorphic to the boundary of an $(n + 1)$-simplex.

1.3. *Definition.* A *combinatorial* n-*manifold* is a simplicial complex M^n such that the link of a simplex s in M^n is either a ball or a sphere of dimension $n - 1 - \dim s$. We denote the link of a simplex s in M^n by $\mathrm{lk}(s, M^n)$.

Remark. Every combinatorial manifold is a topological manifold.

1.4. *Definition.* Let M^n be a combinatorial n-manifold. The union of those simplexes whose links are balls is called the *boundary* of M^n, and we denote it by ∂M^n. We call the set $M^n - \partial M^n$ the *interior* of M^n ($\mathrm{Int}\, M^n$). Notice that ∂M^n is either empty or is a combinatorial $(n - 1)$-manifold without boundary.

1.5. *Definition.* If M^m and M^q are combinatorial manifolds and $M^m \subset M^q$, then this embedding is said to be *proper* if

$$\partial M^m \subset \partial M^q \quad \text{and} \quad \mathrm{Int}\, M^m \subset \mathrm{Int}\, M^q.$$

When these conditions are satisfied, we call the pair (M^q, M^m) a (q, m)-*manifold pair.* When both M^q and M^m are balls, we call the pair a (q, m)-*ball pair;* when both are spheres, we call it a (q, m)-*sphere pair.* The standard (q, m)-*ball pair* is $(\Sigma^{q-m} s, s)$, where s is an m-simplex and Σ^{q-m} denotes the $(q - m)$-fold suspension. The *standard* (q, m)-*sphere pair* is the boundary of the *standard* $(q + 1, m + 1)$-*ball pair.*

The following theorem will be used frequently. A proof can be found in [12, Chapter 4].

1.6. THEOREM. *If* $q - m \geq 3$, *then every* (q, m)-*ball pair* [(q, m)-*sphere pair*] *is piecewise linearly homeomorphic to the standard* (q, m)-*ball pair* [$(q - m)$-*sphere pair*].

1.7. *Definition.* If (M^q, M^m) is a (q, m)-manifold pair, we say that the embedding of M^m in M^q is *locally unknotted* if for each vertex v of M^m, the $(q - 1, m - 1)$-manifold pair $[\mathrm{lk}(v, M^q), \mathrm{lk}(v, M^m)]$ is piecewise linearly homeomorphic to the standard $(q - 1, m - 1)$-ball pair or to the standard $(q - 1, m - 1)$-sphere pair.

1.8. PROPOSITION. *Let* M *be a connected, closed combinatorial* n-*manifold embedded in an* $(n + 1)$-*sphere* S^{n+1}, *and let* C *be the closure of one of the components of* $S^{n+1} - M$. *Then* C *is a combinatorial manifold if and only if the embedding of* M *into* S^{n+1} *is locally unknotted.*

The proof is left to reader.

1.9. LEMMA. *If* (S^n, S^{n-1}) *is an* $(n, n - 1)$-*sphere pair, where the embedding is locally unknotted, then*

$$(S^n, S^{n-1}) \equiv \textit{standard } (n, n-1)\textit{-sphere pair for } n \neq 4, 5.$$

Proof. The conclusion is equivalent to the assertion that the closure of each component of $S^n - S^{n-1}$ is an n-ball. If D is the closure of one of the components of $S^n - S^{n-1}$, then D is topologically homeomorphic to an n-ball [3]. By 1.8, D is a combinatorial manifold. Thus the conclusion follows from [8].

1.10. *Definition.* Let X be a polyhedron, and Y a subpolyhedron. We say there is an *elementary collapse* from X to Y if there exists an n-ball B such that $X = Y \cup B$ and $Y \cap B$ is an (n - 1)-ball in ∂B. We say X *collapses* to Y (notation: $X \searrow Y$) if there exists a finite sequence of elementary collapses going from X to Y.

1.11. *Definition.* Let M be a combinatorial n-manifold, and X a subpolyhedron. A *regular neighborhood* of X in M is a subpolyhedron N of M such that

(i) N is a closed neighborhood of X in M,

(ii) N is a combinatorial n-manifold, and

(iii) $N \searrow X$.

We shall need the following version of the *Regular-Neighborhood Theorem.* A proof can be found in [12, Chapter 3].

1.12. THEOREM. *If* $X \subset \operatorname{Int} M$, *where* M *is a combinatorial manifold and* X *is a subpolyhedron, then any two regular neighborhoods of* X *in* Int M *are ambient isotopic. Moreover, the isotopy can be chosen so that it keeps* $X \cup \partial M$ *fixed.*

1.13. PROPOSITION. *Assume* $j \geq k + 2$ *and*

$$(\partial D^j, S^k) \equiv \textit{standard } (j - 1, k)\textit{-sphere},$$

where D^j *is a* j*-ball. Let* N *be a regular neighborhood of* S^k *in* ∂D^j. *Then there exists a* (k + 1)*-ball* D^{k+1} *properly embedded in* D^j *such that*

(i) $\partial D^{k+1} = S^k$,

(ii) $D^j \searrow N \cup D^{k+1}$.

Proof. By 1.12 it is sufficient to consider the case where

$$D^j = \{(x_1, \cdots, x_j) \in R^j \colon |x_i| \leq 1 \text{ for each } i\},$$

$$S^k = \{(x_1, \cdots, x_j) \in D^j \colon x_i = 0 \text{ for each } i \geq k+2, \text{ and } |x_h| = 1, \text{ for some } h \leq k+1\},$$

$$N = \{(x_1, \cdots, x_j) \in D^j \colon |x_h| = 1 \text{ for some } h \leq k+1\}.$$

Now define

$$D^{k+1} = \{(x_1, \cdots, x_j) \in D^j \colon x_i = 0 \text{ for each } i \geq k+2\}.$$

It is obvious that D^{k+1} is properly embedded in D^j and that D^{k+1} satisfies (i). We shall show that D^{k+1} satisfies (ii), by induction on j.

When j = 2, then k = 0, and the proposition follows immediately.

Assume we have proved the proposition for j - 1. Let D^j, S^k, N, and D^{k+1} be defined as above. If $A \subset D^j$, let

$$A_+ = \{(x_1, \cdots, x_j) \in A \colon x_j \geq 0\},$$

$$A_0 = \{(x_1, \cdots, x_j) \in A \colon x_j = 0\},$$

$$A_- = \{(x_1, \cdots, x_j) \in A \colon x_j \leq 0\}.$$

Now $D_0^j \cup N_\pm$ is a (j - 1)-ball in $\partial D_\pm^j$. Thus $D_\pm^j \searrow D_0^j \cup N_\pm$. Therefore $D^j \searrow D_0^j \cup N$. If $k = j - 2$, we have finished. If $k < j - 2$, then by the induction hypothesis $D_0^j \searrow D^{k+1} \cup N_0$. Thus, by combining the two collapsings, we get $D^j \searrow D_0^j \cup N \searrow (D^{k+1} \cup N)$. This proves 1.13.

1.14. PROPOSITION. *Let* D_0 *and* D_1 *be two* n-*balls and* M *a combinatorial* n-*manifold such that* $M = D_0 \cup D_1$. *Assume further that there exists a* k-*sphere* $S^k \subset M$ *such that*

$$D_0 \cap D_1 = \partial D_0 \cap \partial D_1 = \textit{a regular neighborhood of } S^k \textit{ in } \partial D_i \quad (k \leq n - 4).$$

Then there exists a (k + 1)-*sphere* $S^{k+1} \subset \operatorname{Int} M$ *such that* $M \searrow S^{k+1}$.

Proof. Since $k \leq n - 4$,

$$(\partial D_i, S^k) \equiv \text{standard } (n - 1, k)\text{-sphere pair},$$

by 1.6. Thus there exists a (k + 1)-ball $C_i \subset D_i$ that satisfies the conclusions of 1.13. Define $S^{k+1} = C_0 \cup C_1$. It is clear that $S^{k+1} \subset \operatorname{Int} M$, and

$$M = D_0 \cup D_1 \searrow D_0 \cup C_1 \cup (D_0 \cap D_1) = D_0 \cup C_1 \searrow C_0 \cup C_1 = S^{k+1}.$$

This completes the proof of 1.14.

2. HOMEOMORPHISMS OF $S^p \times S^p$

In this section we consider automorphisms of $H_p(S^p \times S^p, Z)$, and we try to determine under what conditions an automorphism can be induced by a PL homeomorphism of $S^p \times S^p$ onto itself. We study this problem in the differential category, since the maps can be expressed so nicely in that category. Then, using techniques of [7, Chapter 10], we shall arrive at a similar result in the PL category.

2.1. *Notation.* By G we shall denote the group of all 2×2 matrices

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

under matrix multiplication, where the x_i are integers and $|x_1 x_4 - x_2 x_3| = 1$. By G' we shall denote the subgroup of G consisting of all the matrices

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

such that $x_1 \equiv x_4 \pmod 2$ and $x_2 \equiv x_3 \pmod 2$.

2.2. PROPOSITION. G *is generated by*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For a proof, see [10, p. 108].

2.3. PROPOSITION. G' *is generated by*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The proof similar to that of 2.2.

Remark. In order to remain consistent in notation, we let C^n denote the unit n-sphere in euclidean (n + 1)-space R^{n+1} with its usual differential structure, that is,

$$C^n = \left\{ (x_1, \cdots, x_{n+1}) \in R^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

If $x, y \in C^n$, then $x \cdot y$ will denote the inner product of x and y, where x and y are considered as vectors in R^{n+1}. If $n = 1, 3, 7$, then for any $x, y \in C^n$, we denote by xy the *product of* x and y induced by the H-*space structure* of C^n [4].

2.4. *Definition.* If M is the differentiable manifold $C^n \times C^n$ (or the combinatorial manifold $S^n \times S^n$), then a *preferred basis* of M is a basis $\{z_1, z_2\}$ of $H_n(M, Z)$ such that z_1 is represented by $C^n \times v$ (or by $S^n \times v$) and z_2 is represented by $v \times C^n$ (or by $v \times S^n$), where v is any point of C^n (or any vertex of S^n). If f is a topological homeomorphism of M onto itself, and if

$$f_*(z_1) = az_1 + cz_2, \qquad f_*(z_2) = bz_1 + dz_2,$$

we call $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the *matrix induced by* f *with respect to* $\{z_1, z_2\}$.

2.5. PROPOSITION. *Let* n *be odd, and let* $\{z_1, z_2\}$ *be a preferred basis for* $C^n \times C^n$.

(i) *If* $n = 1, 3, 7$, *then for each* $g \in G$ *there exists a diffeomorphism* $f: C^n \times C^n \to C^n \times C^n$ *such that the matrix induced by* f *with respect to* $\{z_1, z_2\}$ *is* g.

(ii) *For each* $g \in G'$, *there exists a diffeomorphism* f *such that* g *is the matrix induced by* f *with respect to* $\{z_1, z_2\}$.

Proof. Let $f_1: C^n \times C^n \to C^n \times C^n$ be defined by

$$f_1(x, y) = (y, x).$$

The matrix induced by f_1 with respect to $\{z_1, z_2\}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let f_2 be defined by

$$f_2(x, y) = (x, \rho y),$$

where ρ is the reflection about the equator. The matrix induced by f_2 is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

To prove (i), it suffices to show, by 2.2, that there exists a diffeomorphism f_3 of $C^n \times C^n$ onto itself such that the matrix induced by f_3 is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Such an f_3 is given by

$$f_3(x, y) = (x, xy).$$

Part (ii) will follow from 2.3, if we can find a diffeomorphism f_4 of $C^n \times C^n$ onto itself whose induced matrix is $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$. Making use of the map defined in [9, p. 14], we define f_4 by

$$f_4(x, y) = (x, y - 2(x \cdot y)x).$$

This completes part (ii), and the proposition is proved.

2.6. LEMMA. *Let* n *be odd, and let* $\{z_1, z_2\}$ *be a preferred basis for* $S^n \times S^n$.

(i) *If* $n = 1, 3, 7$, *then for each* $g \in G$ *there exists a* PL *homeomorphism* $f\colon S^n \times S^n \to S^n \times S^n$ *such that the matrix induced by* f *with respect to* $\{z_1, z_2\}$ *is* g.

(ii) *For each* $g \in G'$, *there exists a* PL *homeomorphism* f *such that the matrix induced by* f *with respect to* $\{z_1, z_2\}$ *is* g.

Proof. Use 2.5 and the fact that any diffeomorphism can be approximated by a PL homeomorphism.

3. THE EMBEDDING OF $S^p \times S^q$ IN S^{p+q+1}

In this section we consider the embedding $f\colon S^p \times S^q \to S^{p+q+1}$, where $p \geq q > 1$. To save space, we write $T = f(S^p \times S^q)$. (This will also be advantageous in the proof of the main lemma in this section, since we leave T fixed and alter the homeomorphism f.) All homology and cohomology groups will have Z, the group of integers, as their coefficient group. Since $H_{p+q}(T) = Z$, it follows that $S^{p+q+1} - T$ has exactly two components. Let C_1 and C_2 denote the closures of these components.

3.1. PROPOSITION. C_1 *and* C_2 *are simply connected.*

Proof. By our assumptions on p and q, T is simply connected. Therefore, by Van Kampen's Theorem, $\pi_1(S^{p+q+1})$ is isomorphic to the free product of $\pi_1(C_1)$ and $\pi_1(C_2)$. The proposition follows from the well-known fact that the free product of two groups is trivial if and only if each of the groups is trivial.

3.2. PROPOSITION. C_1 *and* C_2 *can be indexed so that* $H_*(C_1) = H_*(S^p)$ *and* $H_*(C_2) = H_*(S^q)$.

Proof. Use Alexander duality and the Mayer-Vietoris sequence.

Remark. From now on, we let C_1 be the closure of the component of $S^{p+q+1} - T$ that has the homology groups of a p-sphere, and we denote by i_j the inclusion map of T into C_j. Also, the embedding $f\colon S^p \times S^q \to S^{p+q+1}$ is regarded as a PL homeomorphism of $S^p \times S^q$ onto T.

3.3. LEMMA. *There exists a* PL *homeomorphism* $g: S^p \times S^q \to T$ *such that* $i_1 \circ g(u \times S^q)$ *represents the zero of* $\pi_q(C_1)$.

Proof. By 3.1, 3.2, and the Hurewicz isomorphism theorem, C_1 is $(q-1)$-connected. Thus, the conclusion will follow if we find a PL homeomorphism $g: S^p \times S^q \to T$ such that $i_1 \circ g(u \times S^q)$ represents the zero of $H_q(C_1)$. There are four cases.

Case (i). If $p > q$, then C_1 is q-connected and we may take $g = f$.

Case (ii). If $p = q \equiv 0 \pmod 2$, let $\{z_1, z_2\}$ be a preferred basis for $H_p(S^p \times S^p)$. Let $\{y_1, y_2\}$ be a basis for $H^p(S^p \times S^p)$ such that

$$z_i \frown y_i \neq 0 \quad \text{and} \quad z_i \frown y_j = 0$$

whenever $i \neq j$, where $\frown$ represents the cap product. It is well known that

$$y_i \smile y_i = 0, \quad y_i \smile y_j \neq 0,$$

where $\smile$ represents the cup product. Let s_i be a generator of $H_p(C_i)$, and t_i a generator of $H^p(C_i)$. Then $s_i \frown t_i \neq 0$. Now consider $i_1 \circ f: S^p \times S^p \to C_1$, and let

$$(i_1 \circ f)^*(t_1) = my_1 + ny_2 .$$

It is clear that

$$(i_1 \circ f)^*(t_1 \smile t_1) = (my_1 + ny_2) \smile (my_1 + ny_2).$$

Since $t_1 \smile t_1 = 0$ and p is even,

$$0 = m^2(y_1 \smile y_1) + 2mn(y_1 \smile y_2) + n^2(y_2 \smile y_2) = 2mn(y_1 \smile y_2).$$

Thus either $m = 0$ or $n = 0$. If $n = 0$, then

$$(i_1 \circ f)^*(t_1) = my_1 ,$$

so that

$$[(i_1 \circ f)_*(z_2)] \frown t_1 = (i_1 \circ f)_*(z_2 \frown [(i_1 \circ f)^*(t_1)]) = (i_1 \circ f)_*(z_2 \frown my_1) = 0 .$$

Hence, if $n = 0$, we may let $g = f$.

If $m = 0$, we define a PL homeomorphism h that takes $S^p \times S^p$ onto itself by

$$h(x, y) = (y, x).$$

To complete the proof of Case (ii), we define g by $g = f \circ h$.

Case (iii). If $p = q = 3$ or $p = q = 7$, we use notation from (ii), and we obtain the relations

$$(i_1 \circ f)_*(z_1) = ds_1 , \quad (i_1 \circ f)_*(z_2) = (-b)s_1 .$$

It is clear that $(-b, d) = 1$. Therefore there are integers a and c such that $ad - bc = 1$. Now let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

By 2.6, we have a PL homeomorphism $h: S^p \times S^p \to S^p \times S^p$ that induces γ with respect to $\{z_1, z_2\}$. If we define $g = f \circ h$, then

$$(i_1 \circ g)_*(z_2) = (i_1 \circ f)_* h_*(z_2) = (i_1 \circ f)_*(bz_1 + dz_2) = (bds_1 + d(-b)s_1) = 0.$$

This completes the proof for Case (iii).

Case (iv). Assume that $p = q \equiv 1 \pmod 2$ and $p \neq 3, 7$. Using the notation from case (ii), we again have the identities

$$(i_1 \circ f)_*(z_1) = as_1, \qquad (i_1 \circ f)_*(z_2) = cs_1,$$

$$(i_2 \circ f)_*(z_1) = bs_2, \qquad (i_2 \circ f)_*(z_2) = ds_2.$$

Since the Mayer-Vietoris sequence used in 3.2 gives an isomorphism between $H_p(T)$ and $H_p(C_1) \oplus H_p(C_2)$, we see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Since $\pi_p(C_i) = Z$, let $e_i: S^p \to C_i$ be a *topological* map that represents the generator of $\pi_p(C_i)$. Because S^p and C_i are simply connected and $(e_i)_*$ induces an isomorphism between the homology groups of S^p and C_i, it follows from [11, Theorem 3] that e_i is a homotopy equivalence. Therefore there exists an $h_i: C_i \to S^p$ that is a homotopy equivalence.

Let

w_1 represent $(S^p \times v)$ in $\pi_p(S^p \times S^p)$,

w_2 represent $(u \times S^p)$ in $\pi_p(S^p \times S^p)$,

w represent a generator of $\pi_p(S^p)$.

Then

$$(h_1 \circ i_1 \circ f)_*(w_1) = aw, \qquad (h_1 \circ i_1 \circ f)_*(w_2) = cw,$$

$$(h_2 \circ i_2 \circ f)_*(w_1) = bw, \qquad (h_2 \circ i_2 \circ f)_*(w_2) = dw.$$

We define

$$\phi_j: (S^p \times v \cup u \times S^p) \to S^p$$

by taking $h_j \circ i_j \circ f$ restricted to $(S^p \times v \cup u \times S^p)$. Then ϕ_j determines an element of $\pi_{2p-1}(S^p)$ that is equal to

$$[\pm aw, \pm cw] \quad \text{when } j = 1,$$

$$[\pm bw, \pm dw] \quad \text{when } j = 2$$

(we use the *square-bracket product* of Whitehead [4, p. 8]). Now

$$[\pm aw, \pm cw] = \pm ac[w, w],$$

$$[\pm bw, \pm dw] = \pm bd[w, w].$$

Since S^p is not an H-space [1], $[w, w] \neq 0$, so that it is an element of order 2 [4, p. 18]. Since ϕ_j can be extended to $h_j \circ i_j \circ f\colon S^p \times S^p \to S^p$, we obtain from [4] the relations

$$\pm ac[w, w] = 0, \qquad \pm bd[w, w] = 0.$$

Thus

$$ac \equiv 0 \pmod 2, \qquad bd \equiv 0 \pmod 2.$$

Therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G'$, and hence $\gamma = \begin{pmatrix} d & c \\ -b & -a \end{pmatrix} \in G'$.

By 2.6, there is a PL homeomorphism

$$\theta\colon S^p \times S^p \to S^p \times S^p$$

that induces γ with respect to $\{z_1, z_2\}$. Let $g = f \circ \theta$. Then

$$(i_1 \circ g)_*(z_2) = (i_1 \circ f)_* \theta_*(z_2) = (i_1 \circ f)_*(cz_1 - az_2) = acs_1 - acs_1 = 0.$$

Hence the proof of 3.3 is complete.

4. UNKNOTTING T IN S^{p+q+1}

In this final section, we assume that we have a locally unknotted submanifold $T \subset S^{p+q+1}$ and a PL homeomorphism $g\colon S^p \times S^q \to T$ such that $g(u \times S^q)$ represents the identity in $\pi_q(C_1)$, and that $p \geq q > 1$, $p + q \neq 4$. We shall show that C_1 is a regular neighborhood of some p-sphere embedded in S^{p+q+1}. Then it will follow that T is unknotted in S^{p+q+1}.

4.1. PROPOSITION. *There exists a* $(q + 1)$*-ball* D^{q+1}, *properly embedded in* C_1, *such that* $\partial D^{q+1} = g(u \times S^q)$ *for some vertex* u *of* S^p.

Proof. Consider g restricted to $(u \times S^q)$, where u is any vertex of S^p. This represents the identity element in $\pi_q(C_1)$. Thus we can find a *topological* map

$$h\colon D^{q+1} \to C_1.$$

such that h restricted to ∂D^{q+1} is PL and $h(\partial D^{q+1}) = g(u \times S^q)$. By [12, Theorem 5], we may assume that h is in general position.

Now the conditions of [5, Theorem 1.1] are satisfied. Thus we have a proper embedding

$$s\colon D^{q+1} \to C_1$$

such that $s(\partial D^{q+1}) = g(u \times S^q)$. Hence the proof of 4.1 is complete.

4.2. LEMMA. C_1 *is a regular neighborhood of a* p-*sphere embedded in* S^{p+q+1}.

Proof. Let M be the second derived neighborhood of $s(D^{q+1})$ in C_1. Then M is a $(p+q+1)$-ball. Now $M \cap \partial C_1$ is a regular neighborhood of $g(u \times S^q)$ in ∂C_1. By the construction of g, the set $g^{-1}(M \cap \partial C_1)$ is a regular neighborhood of $(u \times S^q)$ in $S^p \times S^q$. By 1.12, $g^{-1}(M \cap \partial C_1)$ can be assumed to be equal to $D^p \times S^q$, where $u \in \operatorname{Int} D^p$. Thus there exists a PL homeomorphism h, taking $S^p \times S^q$ onto itself, such that

$$h[g^{-1}(M \cap \partial C_1)] = \overline{S^p \times S^q - g^{-1}(M \cap \partial C_1)},$$

$$h \text{ restricted to } \partial[g^{-1}(M \cap \partial C_1)] = \text{identity}.$$

Therefore $g \circ h \circ g^{-1}$ is a PL homeomorphism of ∂C_1 onto itself such that

$$g \circ h \circ g^{-1}[(\partial C_1) \cap M] = \overline{(\partial C_1) - M}$$

and $g \circ h \circ g^{-1}$ is the identity on $\partial[(\partial C_1) \cap M]$. Therefore, ∂M is PL homeomorphic to $\partial(C_1 - M)$. Since M and C_1 are locally unknotted, it follows that $(C_1 - M)$ is locally unknotted in S^{p+q+1}. Therefore $\overline{C_1 - M}$ is a $(p+q+1)$-ball. Since $\overline{(C_1 - M)} \cap M$ is the closure of the complement of a regular neighborhood of an unknotted q-sphere in ∂M, it follows that $\overline{(C_1 - M)} \cap M$ is a regular neighborhood in ∂M of a $(p-1)$-sphere. Therefore, by 1.14, C_1 is a regular neighborhood of a p-sphere. Thus the proof of 4.2 is complete.

4.3. THEOREM. *If* $p \geq q > 1$ *and* $p + q \neq 4$, *then a locally unknotted* $S^p \times S^q$ *unknots in* S^{p+q+1}.

Proof. Let T^α and T^β be two locally flat embeddings of $S^p \times S^q$ in S^{p+q+1}. By 4.2, the closure of one of the components of $S^{p+q+1} - T^\gamma$, say C^γ, is a regular neighborhood of a p-sphere $S^p_\gamma \subset S^{p+q+1}$ $(\gamma = \alpha, \beta)$. By 1.6, we can find a PL homeomorphism $h: S^{p+q+1} \to S^{p+q+1}$ such that

$$h(S^p_\alpha) = S^p_\beta.$$

Now $h(C_\alpha)$ and C_β are regular neighborhoods of S^p_β. Thus, by 1.12, there is a PL homeomorphism $h': S^{p+q+1} \to S^{p+q+1}$ such that

$$h' \circ h(C_\alpha) = (C_\beta).$$

Thus

$$h' \circ h(T_\alpha) = (T_\beta),$$

and the proof of 4.3 is complete.

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