REPRESENTATION RINGS

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1. INTRODUCTION

Let Λ be a ring with unity element. By a Λ-*module* we shall always mean a finitely generated unitary left Λ-module. If C is some category of Λ-modules, we may associate with C an abelian additive group a(C), generated by the set of symbols $\{[M]: M \in C\}$, with relations $[M] = [M'] + [M'']$ whenever $M \cong M' \oplus M''$. From this definition it follows at once that $[M] = [N]$ in a(C) if and only if there exists a module $X \in C$ such that $M \oplus X \cong N \oplus X$.

In particular, suppose that Λ is the group ring RG of a finite group G over an integral domain R. Take C to be the category of R-torsion-free RG-modules, and define multiplication in a(C) by means of

$$[M][N] = [M \otimes_R N] \qquad (M, N \in C).$$

Then a(C) becomes a commutative ring, hereafter denoted by a(RG) and called the *representation ring* of RG. Such rings have been studied in [4] to [7], and in [10].

Now let Z be the ring of rational integers, and let G be a group of order n. Define

$$Z' = \{a/b: a, b \in Z, (b, n) = 1\}.$$

Then Z' is a semilocal ring, useful in the study of indecomposable ZG-modules. The purpose of the present note is to investigate the relationship between the representation rings a(ZG) and a(Z'G), and to settle a conjecture raised at the end of [6].

Two Z-free ZG-modules M, N are said to lie in the same *genus* (notation: $M \vee N$) if and only if $Z' \otimes M \cong Z' \otimes N$. (The original definition of genus, as well as its equivalence with the above definition, may be found in [3]. See also [1, Section 81].)

In this note it will be shown that, as additive groups,

$$a(ZG) \cong b(ZG) \oplus a(Z'G), \tag{1}$$

where b(ZG) is some finite additive group which is an ideal in the ring a(ZG). Explicitly,

$$b(ZG) = \{[F] - [P]: F = \text{free ZG-module}, P \vee F\}. \tag{2}$$

We easily deduce that

$$b(ZG) = \{[ZG] - [P]: P \vee ZG\}, \tag{3}$$

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from which the finiteness of b(ZG) is an immediate consequence. As in [13], one shows that

(4) $$\{b(ZG)\}^2 = 0 \quad \text{in } a(ZG).$$

Finally, we remark that b(ZG) is a quotient of the reduced projective class group c(ZG) defined in [8] and [11].

2. MODULES OVER ORDERS

Throughout this section, R is a Dedekind ring with quotient field K, and $\mathfrak{A}$ is a finite-dimensional separable algebra over K. Let Λ be an R-order in $\mathfrak{A}$, and let $C_f(\Lambda)$ be the category of R-torsion-free Λ-modules. For a prime ideal p in R, denote by R_p the localization of R at p, and by R_p^* the p-adic completion of R_p. We set

$$\Lambda_p = R_p \otimes_R \Lambda, \qquad \Lambda_p^* = R_p^* \otimes_R \Lambda.$$

Finally, let $\Lambda^{(k)}$ denote the direct sum of k copies of Λ.

As was shown by Higman [2] (see [1, (75.11)]), there exists a nonzero ideal $i(\Lambda)$ in R such that

$$i(\Lambda) \cdot \mathrm{Ext}_\Lambda^1(A, B) = 0$$

for all Λ-modules A and B, provided only that A is R-torsion-free. Now define

$$R' = \bigcap_{p \supset i(\Lambda)} R_p, \qquad \Lambda' = R' \otimes_R \Lambda.$$

Two modules M, N $\in C_f(\Lambda)$ are in the *same genus* (notation: $M \vee N$) if $R_p^* M \cong R_p^* N$ for all p. As in [3] or [1, Section 81], $M \vee N$ if and only if $R'M \cong R'N$. (In case $i(\Lambda) = R$, the ring R' is chosen to the field K.) Equivalently, $M \vee N$ if and only if for each ideal q in R there exists a Λ-monomorphism ϕ: M $\to$ N such that

$$q + \mathrm{ann}(N/\phi M) = R.$$

Here,

$$\mathrm{ann}(N/\phi M) = \{\alpha \in R: \alpha \cdot N \subset \phi M\}.$$

Now let $a(\Lambda)$ be the additive group associated with the category $C_f(\Lambda)$, and define $a(\Lambda')$, $a(\Lambda_p^*)$ analogously. There is an additive homomorphism

$$\tau: a(\Lambda') \to \prod_{p \supset i(\Lambda)} a(\Lambda_p^*),$$

defined by

$$\tau[M'] = \prod_{p \supset i(\Lambda)} [R_p^* M'] \qquad (M' \in C_f(\Lambda')).$$

Since the Krull-Schmidt theorem holds for Λ_p^*-modules (see [1, (76.25)]), it follows that for each p the additive group $a(\Lambda_p^*)$ is Z-free, with Z-basis

$$\{[Y]: Y \in C_f(\Lambda_p^*),\ Y \text{ indecomposable}\}.$$

Furthermore, the results in [3] show that τ is monic. Thus $a(\Lambda')$ is embedded in a finite direct product of Z-free Z-modules, and therefore $a(\Lambda')$ is also Z-free.

(In the trivial case where $i(\Lambda) = R$, the above discussion breaks down. However, in this case $R' = K$, and $\Lambda' = \mathfrak{A}$, so it is clear that $a(\Lambda')$ is Z-free.)

The next result is a special case of a general theorem due to Roiter [9], and we give a simple proof for this case.

LEMMA. *Let* $M, N \in C_f(\Lambda)$, *and suppose that* $M \vee N$. *Then there exist a positive integer* k *and a module* $P \in C_f(\Lambda)$ *such that*

$$M \oplus \Lambda^{(k)} \cong N \oplus P.$$

Furthermore, $P \vee \Lambda^{(k)}$, *and* P *is a projective* Λ*-module.*

Proof. Since $M \vee N$, there exists a Λ-monomorphism $\phi: M \to N$ such that

$$i(\Lambda) + \operatorname{ann}(N/\phi M) = R.$$

Hence there is an exact sequence of Λ-modules:

$$0 \to M \to N \xrightarrow{h} T \to 0,$$

where T is an R-torsion Λ-module such that $i(\Lambda) + \operatorname{ann} T = R$. Let us write

$$1 = \alpha + \beta \qquad (\alpha \in i(\Lambda),\ \beta \in \operatorname{ann} T).$$

Then $h = (1 - \beta)h = \alpha h$.

Now let

$$0 \to B \to C \xrightarrow{u} T \to 0$$

be any exact sequence of Λ-modules. Then there is an exact sequence of R-modules:

$$\operatorname{Hom}_\Lambda(N, C) \xrightarrow{u^*} \operatorname{Hom}_\Lambda(N, T) \xrightarrow{\delta} \operatorname{Ext}^1_\Lambda(N, B).$$

Since $\alpha \in i(\Lambda)$, we see that

$$\delta(h) = \delta(\alpha h) = \alpha\delta(h) = 0,$$

and therefore h lies in the image of u^*. We have thus shown that each diagram

$$\begin{array}{ccccc} & & N & & \\ & & \downarrow{\scriptstyle h} & & \\ C & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

whose bottom row is exact can be completed to a commutative diagram

$$\begin{array}{ccccc} & & N & & \\ & \swarrow & \downarrow h & & \\ C & \longrightarrow & T & \longrightarrow & 0. \end{array}$$

In particular, choose a free module $\Lambda^{(k)}$ mapping onto T, and let P be the kernel of the map. Then there exists a map $N \to \Lambda^{(k)}$ making the following diagram commutative:

$$\begin{array}{ccccc} N & \xrightarrow{h} & T & \longrightarrow & 0 \\ \downarrow & & \downarrow 1 & & \\ \Lambda^{(k)} & \longrightarrow & T & \longrightarrow & 0. \end{array}$$

Since the same argument also yields a map $\Lambda^{(k)} \to N$, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & T & \longrightarrow & 0 \\ & & & & \downarrow\uparrow & & \updownarrow 1 & & \\ 0 & \longrightarrow & P & \longrightarrow & \Lambda^{(k)} & \longrightarrow & T & \longrightarrow & 0. \end{array}$$

It follows then from the method of proof of Schanuel's lemma (see [12], for example) that

$$M \oplus \Lambda^{(k)} \cong N \oplus P,$$

as desired.

The above isomorphism immediately implies that $P \vee \Lambda^{(k)}$, since for each p we have

$$R_p^* M \oplus \Lambda_p^{*(k)} \cong R_p^* N \oplus R_p^* P,$$

and therefore (by virtue of the Krull-Schmidt theorem for Λ_p^*-modules) we may conclude that

$$\Lambda_p^{*(k)} \cong R_p^* P.$$

This also shows that P is Λ-projective ([1, (77.1)]).

COROLLARY. *Let* $M \vee N$, *where* $M, N \in C_f(\Lambda)$. *Then there exist a positive integer* k *and a projective* Λ*-module* P_1 *in the same genus as* Λ, *such that*

$$M \oplus \Lambda^{(k)} \cong N \oplus P_1 \oplus \Lambda^{(k-1)}.$$

Proof. Let P and k be as in the preceding lemma. Since $P \vee \Lambda^{(k)}$, the method of Swan [11] (see [1, (78.5)]) can be used to show that

$$P \cong \Lambda^{(k-1)} \oplus P_1$$

for some projective Λ-module P_1 in the same genus as Λ.

Let us now define a mapping μ: $a(\Lambda) \to a(\Lambda')$ by setting

$$\mu[M] = [R'M] \qquad (M \in C_f(\Lambda)).$$

Then μ is well-defined and is an additive homomorphism. It is easily seen that μ is an epimorphism (for example, see [13] or [1, (73.5)]). If we denote the kernel of μ by $b(\Lambda)$, then there is an exact sequence of additive groups:

$$0 \to b(\Lambda) \to a(\Lambda) \to a(\Lambda') \to 0.$$

Since $a(\Lambda')$ is Z-free, the sequence splits, and thus

$$a(\Lambda) \cong b(\Lambda) \oplus a(\Lambda')$$

as additive groups.

Furthermore, let $[M] - [N] \in b(\Lambda)$, where $M, N \in C_f(\Lambda)$. Then $[R'M] = [R'N]$ in $a(\Lambda')$, and so for each p, $[R_p^* M] = [R_p^* N]$ in $a(\Lambda_p^*)$. Since the Krull-Schmidt theorem holds for Λ_p^*-modules, the last equality implies that $R_p^* M \cong R_p^* N$ for each p, and thus $M \vee N$. We may then apply the preceding corollary, obtaining the relation

$$[M] - [N] = [P_1] - [\Lambda] \quad \text{in } a(\Lambda).$$

We have therefore shown that

$$b(\Lambda) = \{[\Lambda] - [P_1]: P_1 \vee \Lambda\}.$$

Suppose now that the number of ideal classes in R is finite, and that for each (nonzero) prime ideal p of R the residue class field R/p is finite. Then the Jordan-Zassenhaus theorem is applicable (see [1, Section 79]), and so the number of isomorphism classes of Λ-modules P_1 in the same genus as Λ is finite. Thus, in this case, the group $b(\Lambda)$ is finite.

3. MODULES OVER GROUP RINGS

We now take $\Lambda = RG$, where G is a finite group of order n, and R is the ring of all algebraic integers in some algebraic number field K. As shown in [2] (see [1, Section 75]), the Higman ideal i(RG) is precisely the principal ideal nR.

As we remarked in the introduction, a(RG) and a(R'G) are rings, and it is obvious that the maps μ and τ of Section 2 are ring homomorphisms. Thus, b(RG) is not only a finite additive group, but it is also an ideal in a(RG).

We have now established formulas (1) to (3) of Section 1, and we proceed to sketch the proof of (4), as found in [13]. We shall show that $\{b(RG)\}^2 = 0$ in a(RG). Let

$$M_i, N_i \in C_f(RG), \quad M_i \vee N_i \quad (i = 1, 2).$$

There exist exact sequences

$$0 \to M_i \to N_i \to T_i \to 0 \quad (i = 1, 2),$$

with T_1 and T_2 R-torsion RG-modules such that

(5) $$\text{ann } T_1 + \text{ann } T_2 = R, \quad \text{ann } T_i + nR = R \quad (i = 1, 2).$$

Thus, there are exact sequences of RG-modules:

(6) $$0 \to M_1 \otimes M_2 \to N_1 \otimes M_2 \to T_1 \otimes M_2 \to 0,$$

(7) $$0 \to M_1 \otimes N_2 \to N_1 \otimes N_2 \to T_1 \otimes N_2 \to 0,$$

and also an exact sequence of R-modules:

(8) $$\mathrm{Tor}_1^R(T_1, T_2) \to T_1 \otimes M_2 \to T_1 \otimes N_2 \to T_1 \otimes T_2.$$

The first and last terms in (8) are both zero, by virtue of (5). Therefore $T_1 \otimes M_2 \cong T_1 \otimes N_2$. These modules are R-torsion RG-modules whose annihilator is relatively prime to nR. Applying the method in Section 2, we may thus deduce from (6) and (7) that

$$M_1 \otimes M_2 \oplus N_1 \otimes N_2 \cong M_1 \otimes N_2 \oplus N_1 \otimes M_2.$$

This shows that

$$([M_1] - [N_1])([M_2] - [N_2]) = 0,$$

and establishes that $\{b(RG)\}^2 = 0$.

To conclude, let us investigate the relationship between b(RG) and the reduced projective class group c(RG) defined in [8] and [11]. According to [8],

$$c(RG) = \{[M] - [N]: M, N \text{ projective RG-modules, } KM \cong KN\}.$$

Further, $[M] = [N]$ in c(RG) if and only if there exists a free RG-module F such that $M \oplus F \cong N \oplus F$. However, it was proved in [11] (see [1, Section 78]) that if M is any projective RG-module, then there exists a free RG-module F such that $M \vee F$, and thus automatically $KM \cong KF$. Conversely, an RG-module in the same genus as a free module must be projective. Therefore

$$c(RG) = \{[F] - [M]: F = \text{free RG-module, } M \vee F\}.$$

An easy argument (see [11]) then shows that

$$c(RG) = \{[RG] - [P]: P \vee RG\}.$$

From the preceding discussion, we conclude at once that the map $\lambda: c(RG) \to b(RG)$, defined by letting the expression $[RG] - [P]$ map onto itself, is a ring epimorphism. However, λ need not be an isomorphism. Indeed, if $P \vee RG$, then $[RG] - [P] = 0$ in c(RG) if and only if there exists a free module F such that $RG \oplus F \cong P \oplus F$. On the other hand, $[RG] - [P] = 0$ in b(RG) if and only if the isomorphism $RG \oplus X \cong P \oplus X$ holds for some R-torsion-free RG-module X. It seems difficult, however, to give a specific example in which λ is not a monomorphism.

In order to determine the ring structure of a(RG), it is necessary to give first the structure of $a(R_p^* G)$ for each prime ideal p. Once this is known, we may regard the ring $a(R'G)$ as known, and we can try to describe its action on the additive group b(RG). This is likely to be a difficult question, since the corresponding problem for Grothendieck groups is already quite complicated (see [13]).

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