A COVERING-SPACE APPROACH TO RESIDUAL PROPERTIES OF GROUPS

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To R. L. Wilder, on his seventieth birthday.

INTRODUCTION

A group G is said to have a property P residually, or to be "residually P" if and only if for each $x \in G$ there is a normal subgroup V in G such that $x \notin V$ and G/V has the property P. This holds if and only if the intersection J(P) of the family $\mathscr{M}(P)$ of all such subgroups V is trivial. Algebraists have developed certain methods for proving that $J(P) = \{1\}$ for certain properties P; a good example is the theorem of Baumslag [5, p. 414], where P is the property of being finite. However, when G arises as the fundamental group of a path-connected topological space X (say a surface), it is of interest to look at the family of associated covering spaces X_A ($A \in \mathscr{M}(P)$), which form an inverse system with inverse limit X_*. Comparing this with X_J (where $J = J(P)$), we obtain our main result (Theorem 1 below), on the assumption that $\mathscr{M}$ has a linearly ordered cofinal subset relative to its ordering by inclusion; this theorem states that *there is a natural monomorphism* $H_1(X_J) \to H_1(X_*)$ *of Čech homology groups with compact carriers and arbitrary coefficients.*

Since X_* is an inverse limit, it is often fairly easy to compute $H_1(X_*)$; and if X (and hence X_J) is (say) locally contractible, then $H_1(X_J)$ is isomorphic to the singular homology group $H_1 S(X_J)$. The latter group, with coefficients in the group $\underset{\sim}{Z}$ of integers, is the abelianizer of J; therefore

the abelianizer of J *is isomorphic to a subgroup of* $H_1(X_*, \underset{\sim}{Z})$.

Thus, for example, if G is known not to possess perfect subgroups, then J is trivial if $H_1(X_*, \underset{\sim}{Z})$ is zero. Some applications are given in the last section (6) of this paper; for example, we show that the fundamental group of a surface is residually finite. The only previously known proof of this depends on the theory of Fuchsian groups (as a deduction from the theorem of Baumslag, quoted above). In Section 5, we prove our main theorem, using the somewhat technical Lemma 3 from Section 3. The remaining sections establish notation and gather standard requisite ideas about covering spaces, inverse limits, and Čech homology.

1. COVERING SPACES

We shall be interested in groups of the form $G = \pi_1(X, o)$, where X denotes a path-connected, LC^1, locally compact metric space with base-point o. Let $\mathscr{L}$ denote the lattice of all subgroups of G. Then with each $A, B \in \mathscr{L}$ such that $A \subseteq B$ there are associated based covering spaces and projection maps

(1.1) $$p_{AB}: (X_A, o_A) \to (X_B, o_B);$$

Received October 31, 1966.

we collect here some general relevant facts concerning them (see [4, Chapter 6]). For each map p_{AB}, the induced homomorphism

$$\pi_{AB}: \pi_1(X_A, o_A) \to \pi_1(X_B, o_B) \tag{1.2}$$

is a monomorphism. Moreover, if $B = G$, then π_{AG} is an isomorphism of $\pi_1(X_A, o_A)$ onto $A \subseteq G$. It is convenient to think of X_A as the quotient space E/R_A of the space E of all o-based paths $\lambda: (I, 0) \to (X, o)$ determined by an equivalence relation R_A which we now describe. We write $\lambda R_A \mu$, whenever $\lambda(1) = \mu(1)$ and the loop formed by the composition $\lambda * \mu^{-1}$ represents an element of A in $\pi_1(X, o)$; in particular, $\lambda R_A \mu$ if λ is homotopic to μ with fixed end-points. Thus, if X_A is taken to be E/R_A, the map p_{AB} is the inclusion map $[\lambda]_A \to [\lambda]_B$ of cosets (since $\lambda R_A \mu$ implies $\lambda R_B \mu$); hence it is a surjection. Further, if $A \subseteq B \subseteq C$ in $\mathscr{L}$, then

$$p_{BC} \circ p_{AB} = p_{AC}. \tag{1.3}$$

Each X_A has base-point $o_A = [o]_A$, the class of the constant map $X \to \{o\}$. Henceforth, we identify X_G with X, using the bijection $[\lambda]_G \to \lambda(1)$ under which o_G identifies with o; we then identify the map p_{AG} with the end-point map $[\lambda]_A \to \lambda(1)$.

The topology of X_A is the "canonical-neighbourhood" topology, described as follows. Each $x \in X$ has arbitrarily small "canonical neighbourhoods" U which are path-connected with compact closure, and such that every loop in U is contractible in X to a point. A basis of neighbourhoods of $[\lambda]_A \in X_A$ is the family $\{U'\}$, where each U' corresponds to a canonical neighbourhood U of $\lambda(1)$ in X, and where

$$U' = \{[\lambda * \varepsilon]_A\}, \tag{1.4}$$

as ε runs through all paths in U starting at $\lambda(1)$. It follows that $p_{AG} \mid U'$ is a homeomorphism onto U and that the components of $p_{AG}^{-1}(U)$ are all of the form U', one for each point in $p_{AG}^{-1}(\lambda(1))$. Each such U' is called "canonical" in X_A, and by (1.4), $p_{AB} \mid U'$ is a homeomorphism onto the canonical set $p_{AB}(U')$ in X_B. Observe that U in X has these properties simultaneously *for all* $A \in \mathscr{L}$, again by (1.4).

Next suppose that A is normal in B. Then the cosets bA of the group B/A act as homeomorphisms of X_A by the rule

$$bA \cdot [\lambda]_A = [\beta * \lambda]_A, \tag{1.5}$$

where the dot denotes the action of bA, and β is any loop representing $b \in \pi_1(X, o)$. The coset bA, regarded as a homeomorphism $X_A \to X_A$, is without fixed-points (unless it is the identity) and it is a covering transformation, in the sense that

$$p_{AB} \circ bA = p_{AB}, \tag{1.6}$$

because $[\beta * \lambda]_B = [\lambda]_B$ (by (1.5) and the definition of R_B). Indeed, for each $x \in X_B$, $p_{AB}^{-1}(x)$ is the orbit $B/A \cdot y$ of any $y \in p_{AB}^{-1}(x)$, and it is a discrete set. More generally, if $A \subseteq B \subseteq C$ and A and B are normal in C, then for each $c \in C$ we have the commutative diagram

$$\begin{array}{ccc} X_A & \xrightarrow{cA} & X_A \\ {\scriptstyle p_{AB}}\downarrow & & \downarrow{\scriptstyle p_{AB}} \\ X_B & \xrightarrow{cB} & X_B \end{array} .$$

It is useful to introduce an invariant metric in X_A, as follows. We assumed above that X has a metric, say ρ. Thus E also has the usual sup metric d, whence X_A has metric d_A given by

$$(1.8) \qquad d_A([\lambda]_A, [\mu]_A) = \inf\{d(\lambda', \mu') \mid \lambda' \in [\lambda]_A, \ \mu' \in [\mu]_A\}.$$

(If λ', μ' are sufficiently close in E, with the same end-points, then they are homotopic since X is LC^1; thus $d_A([\lambda]_A, [\mu]_A) = 0$ if and only if $[\lambda]_A = [\mu]_A$.) Observe that since $[\lambda]_A \subseteq [\lambda]_B$,

$$d_A([\lambda]_A, [\mu]_A) \geq d_B([\lambda]_B, [\mu]_B) \geq \rho(\lambda(1), \mu(1)),$$

while if λ, μ lie in a subset of X of diameter h, then the left-hand member is at most 2h. From this it follows that d_G is equivalent to ρ on X_G, and that the canonical-neighbourhood topology on X_A is the d_A-topology. And from (1.5) and (1.8) we see at once that

$$d_A(bA \cdot [\lambda]_A, bA \cdot [\mu]_A) = d_A([\lambda]_A, [\mu]_A),$$

whence bA acts as an *isometry:* d_A *is invariant under the action of* B/A.

2. OPEN COVERINGS

We now need some technical lemmas about open coverings in X_A; they will be used later in connection with the Čech homology theory of X_A.

Let K be a compact subset of X_A, and let Cov K denote the family of all finite coverings of K, of the form $\mathscr{U} = \{U_1, \cdots, U_n\}$, where each U_i is *open in* X_A *and meets* K. We now assume that A is normal in $G = \pi_1(X_G, o_G)$, so that $\Gamma = G/A$ acts on X_A as in (1.5). Let $\Gamma\mathscr{U} \wedge K$ denote the family of all sets $g \cdot U_j$, where $g \in \Gamma$, $U_j \in \mathscr{U}$, and $g \cdot U_j$ meets K. Since K is compact and g is a homeomorphism with discrete orbits, there are only finitely many $g \in \Gamma$ such that K is met by the Γ-iterates $g \cdot U_j$ of a fixed U_j. Thus $\Gamma\mathscr{U} \wedge K \in$ Cov K. We call $\Gamma\mathscr{U} \wedge K$ the Γ-*iterate of* $\mathscr{U}$.

LEMMA 1. *Every* $\mathscr{U} \in$ Cov K *is refined by the* Γ-*iterate of some* $\mathscr{V} \in$ Cov K.

Proof. In the d_A-metric, $\mathscr{U}$ has a Lebesgue number $u > 0$, whence $\mathscr{U}$ is refined by some $\mathscr{V} \in$ Cov K such that each $V \in \mathscr{V}$ has diameter less than u. But then, by invariance of the metric, each member of $\Gamma\mathscr{V} \wedge K$ also has diameter less than u, whence it refines $\mathscr{U}$, as required.

LEMMA 2. *If* $\mathscr{U} \in$ Cov K *is sufficiently fine, and* U, V *are two members of* $\mathscr{U}$, *then* U *does not meet* $g \cdot V$ *for more than one* $g \in \Gamma$.

Proof. Each point $x \in K$ has a neighbourhood W_x such that $W_x \cap gW_x = \emptyset$ for all $g \neq 1$ in Γ. Let $\mathscr{W}$ be a finite covering of K by such sets W_x, and let $\mathscr{U} \in$ Cov K be so fine that each member of $\mathscr{U}$ has diameter less than half the Lebesgue number of $\mathscr{W}$. If now $U, V \in \mathscr{U}$ and $U \cap V \neq \emptyset$, then $V \subseteq \text{St}\, U \subseteq W_x$, say for some $W_x \in \mathscr{W}$. Hence, for each $g \in \Gamma$ $(g \neq 1)$,

$$g \cdot V \subseteq g \cdot W_x \quad \text{and} \quad W_x \cap g \cdot W_x = \emptyset.$$

Thus $U \cap g\cdot V \subseteq \mathrm{St}\, U \cap g\cdot V \subseteq W_x \cap g\cdot V = \emptyset$. Therefore $\mathscr{U}$ is the "sufficiently fine" member of Cov K we require.

In connection with the Čech theory, let us denote by NF the nerve of a finite family F of sets; then (see [7, Chapter V]) NF is a simplicial complex whose vertices are the elements of F, and whose k-simplices are those subsets of F which consist of $k+1$ elements whose intersection meets K. For any subset P of X_A, write

$$|\Gamma\cdot P| = \bigcup_{g\in\Gamma} g\cdot P;$$

if F is a family of subsets of X_A, write

$$(2.1) \qquad |\Gamma\cdot F| = \{\, |\Gamma\cdot P| \;\big|\; P \in F \}.$$

Now let K be a compact subset of X_A, and let $\mathscr{W} \in$ Cov K be a Γ-iterate as in Lemma 1, so fine that it behaves like $\mathscr{U}$ in Lemma 2.

LEMMA 3. *Let* $f\colon N\mathscr{W} \to N|\Gamma\cdot\mathscr{W}|$ *denote the simplicial map induced by the function* $W \longmapsto |\Gamma\cdot\mathscr{W}|$ $(W \in \mathscr{W})$. *Suppose that* z *is a* 1-*cycle on* $N\mathscr{W}$ *such that* $fz \sim 0$ *on* $N|\Gamma\cdot\mathscr{W}|$. *Then* $z \sim 0$ *on* $N\mathscr{W}$.

Proof. By hypothesis, $fz = \partial c$, where c is a 2-chain of $N|\Gamma\cdot\mathscr{W}|$. We shall use induction on the number n of 2-simplexes of c, by showing that $z \sim z_o$ on $N\mathscr{W}$, where fz_o bounds a 2-chain c_o of $N|\Gamma\cdot\mathscr{W}|$ having $n-1$ 2-simplexes. Thus, if $z = k\cdot e + z'$, where e is an edge of z, with multiplicity k and with vertices $U, V \in \mathscr{W}$ (so that e does not appear in the 1-chain z'), then e must be an edge of a 2-simplex uvw of $N|\Gamma\cdot\mathscr{W}|$. Here, we may choose the notation so that

$$u = |\Gamma\cdot U|, \qquad v = |\Gamma\cdot V|, \qquad w = |\Gamma\cdot W|$$

for some $W \in \mathscr{W}$. Since UV and uvw are simplices, it follows from the definition of N that

$$\text{(i) } U \cap V \cap K \neq \emptyset, \qquad \text{(ii) } u \cap v \cap w \cap K \neq \emptyset;$$

therefore, by (ii), there exist elements $a, b \in \Gamma$ such that $U \cap a\cdot V \cap b\cdot W \cap K \neq \emptyset$. But then $U \cap a\cdot V \cap K \neq \emptyset$; together with (i) and the choice of $\mathscr{W}$, Lemma 2 implies that $a = 1 \in \Gamma$. Now $W \in \mathscr{W}$ and $c\cdot W \cap K \neq \emptyset$, while $\mathscr{W}$ is a Γ-iterate; therefore $c\cdot W \in \mathscr{W}$. Hence U, V, and $c\cdot W$ are the vertices of a 2-simplex s of $N\mathscr{W}$, so that

$$z \sim z_1 = z \pm k\cdot\partial s \quad \text{on } N\mathscr{W},$$

where that sign is chosen (depending on the orientation of s) which eliminates e from z_1. We then obtain the relation

$$fz \sim fz_1 \pm k\cdot\partial(uvw) \quad \text{on } N|\Gamma\cdot\mathscr{W}|.$$

Similarly, by eliminating any other edges e' in z for which $fe' = fe = uv$, we obtain a 1-cycle z_o on $N\mathscr{W}$ such that $z \sim z_o$ on $N\mathscr{W}$ while $fz \sim fz_o + \partial\left(\sum k_i t_i\right)$, where the k_i are integers and the t_i are distinct 2-simplices of $N|\Gamma\cdot\mathscr{W}|$ having some fe' as an edge. Hence $fz_o = \partial c_o$ on $N|\Gamma\cdot\mathscr{W}|$, where c_o is c minus the t_i, so that c_o has fewer 2-simplexes than c. We can therefore apply induction on n, provided we can settle the case $n = 1$. In this case, however, it is easily seen that z has the form $\sum k_i \partial s_i$, where the k_i are integers, while each s_i is a 2-simplex of $N\mathscr{W}$ with $f(s_i) = uvw$ (like s above) and is also an iterate $g_i \cdot s_o$ of s_o ($g_i \in \Gamma$). Thus $z \sim 0$ on $N\mathscr{W}$ in every case, and Lemma 3 is proved.

3. INVERSE LIMITS

The lattice $\mathscr{L}$ of subgroups of $G = \pi_1(X, o)$ is ordered by inclusion. Hence, for any subset $\mathscr{M} \subseteq \mathscr{L}$ (not necessarily cofinal), the system $\{(X_A, o_A), p_{AB}\}_{\mathscr{M}}$ of spaces and maps (1.1) with A, B $\in \mathscr{M}$, forms an inverse system. Ignoring the topology for the moment, we note that the system has an inverse limit

(3.1) $$(X_\infty, o_\infty)_{\mathscr{M}} = \text{Inv Lim}\,\{(X_A, o_A), p_{AB}\}_{\mathscr{M}}$$

with projection functions forming a commutative triangle

(3.2) $$\begin{array}{ccc} & \overset{q_A}{\nearrow} & (X_A, o_A) \\ (X_\infty, o_\infty) & & \big\downarrow p_{AB} \\ & \underset{q_B}{\searrow} & (X_B, o_B) \end{array}\ .$$

The elements of X_∞ are "threads" $\langle x_A \rangle_{A \in \mathscr{M}}$; that is to say, $\langle x_A \rangle = \langle x_A \rangle_{A \in \mathscr{M}}$ is a point in the Cartesian product $\prod \{X_A \mid A \in \mathscr{M}\}$ such that $p_{AB}(x_A) = x_B$ for all pairs $A \subseteq B$ in $\mathscr{M}$. Therefore $q_A(\langle x_A \rangle) = x_A$. We shall normally omit the letter $\mathscr{M}$, since it is fixed throughout most of the discussion.

Suppose now that every $A \in \mathscr{M}$ is normal in $G = \pi_1(X_G, o_G)$. We are especially interested in the subgroup

(3.3) $$J = J(\mathscr{M}) = \bigcap \{A \mid A \in \mathscr{M}\},$$

which does not necessarily lie in $\mathscr{M}$ but is normal in G. We use (1.7) to define an action of the group G/J on X_∞ by the rule

(3.4) $$gJ \cdot \langle x_A \rangle = \langle gA \cdot x_A \rangle ;$$

standard checks show that this action is independent of the various choices and that the right-hand member lies in X_∞, while $gJ\colon X_\infty \to X_\infty$ is one-to-one and has no fixed points (since each gA acts on X_A without fixed points). Now G/J also acts on the covering space X_J, and there is a function

(3.5) $$\phi\colon X_J \to X_\infty$$

given by $\phi(x) = \langle p_{JA}(x) \rangle$. Then for each $gJ \in G/J$ (regarded as a transformation) the following equations are implied by (1.7):

$$gJ(\phi(x)) = \langle gA \cdot p_{JA}(x) \rangle = \langle p_{JA} \circ gJ(x) \rangle = \phi(gJ(x)).$$

Hence ϕ commutes with the action of G/J in the sense that

(3.6) $$gJ \circ \phi = \phi \circ gJ\,.$$

LEMMA 4. *ϕ is one-to-one.*

Proof. If $\phi x = \phi y$, then $y \in p_{JA}^{-1}(x)$ for each $A \in \mathscr{M}$; therefore $aJ \cdot y = x$ for some $a \in A$. But if $A \subseteq B$ in $\mathscr{M}$, then $A/J \subseteq B/J \subseteq G/J$; therefore the transformations aJ and bJ in G/J agree at y. Hence $aJ = bJ$, since G/J acts without fixed points; hence $b \in A \supseteq J$, for all $A \subseteq B$; thus $b \in J$ and $y = x$. This proves that ϕ is one-to-one, as required.

The group G/J can be identified with a subgroup of

$$G_\infty = \text{Inv Lim}\,\{G/A,\ j_{AB}\}_{\mathscr{M}}, \tag{3.7}$$

where j_{AB}: $G/A \to G/B$ is the natural homomorphism when $A \subseteq B$ in $\mathscr{M}$; we identify gJ with the thread $\langle gA \rangle$. Then the action of G/J extends to an action of G_∞ on X_∞ by the rule

$$\langle g_A A \rangle \cdot \langle x_A \rangle = \langle g_A A \cdot x_A \rangle,$$

and it is easily verified that G_∞ acts without fixed points.

Let $u \in X_G$ and let q_G: $X_\infty \to X_G$ play the role of q_A in (3.2). Then for each $u \in X_G$ and each $x \in q_G^{-1}(u)$, we have $p_{AG}^{-1}(u) = |G/A \cdot x_A|$ (see (1.6)). Hence we may assert

$$q_G^{-1}(u) = |G_\infty \cdot x|. \tag{3.8}$$

4. TWO TOPOLOGIES

We intend to put two topologies on X_∞, one the inverse limit topology, and the other a topology with more open sets. Our aim, as explained in the Introduction, is to show that ϕ induces a monomorphism of the 1-dimensional singular homology of X_J in a Čech group of the inverse limit. Recall from (1.4) that each X_A has the "canonical-neighbourhood" topology, where, if U is a canonical neighbourhood in X_G with q as in (3.2), then

$$q_G^{-1}(U) = \text{Inv Lim}\,\{p_{AG}^{-1}(U),\ p_{AB}\}_{\mathscr{M}}; \tag{4.1}$$

here (and often below) p_{AB} denotes the appropriate restriction of the map p_{AB} of (1.1). We can verify this equation by using the fact that if $A \subseteq B$ in $\mathscr{M}$, then by (1.3)

$$p_{AG}^{-1}(x) = p_{AB}^{-1}(p_{BG}^{-1}(x)).$$

Thus the set of components of $p_{AG}^{-1}(U)$ is the union of the sets $p_{AB}^{-1}(U_\alpha)$ $(\alpha \in p_{BG}^{-1}(x))$.

In the case where each $A \in \mathscr{M}$ is normal in G,

$$p_{AG}^{-1}(U) = |G/A \cdot U_A| \tag{4.2}$$

for each component U_A of $p_{AG}^{-1}(U)$. Thus $p_{AG} \mid U_A$ is a homeomorphism onto U, and $|G/A \cdot U_A|$ is the orbit of U_A—a disjoint union of G/A-iterates of U_A. For each $A \in \mathscr{M}$ and each $x = \langle x_A \rangle \in X_\infty$ there is a unique U_A containing x_A; hence, if $x_G \in U$, we form the "special" set

(4.3) $$\mathfrak{U}_x = \text{Inv Lim}\,\{U_A, p_{AB}\}_{\mathcal{M}} \subseteq X_\infty .$$

By (3.4), (3.8), and (4.1), it has the property

(4.4) $$q_G^{-1}(U) = |G_\infty \cdot \mathfrak{U}_x| .$$

To define the two topologies for X_∞ mentioned above, we choose certain subsets of the sets $q_G^{-1}(U)$ as bases of open sets, where U varies in X_G. Consider first the usual inverse limit topology on X_∞, and let the resulting topological space be denoted by X_*. The topology is that induced by the inclusion of X_∞ in the Cartesian product $\prod\{X_A \mid A \in \mathcal{M}\}$; hence a basis of neighbourhoods of $\langle x_A \rangle \in X_*$ is the family of sets $\underset{\sim}{W}$ such that

(4.5) $$\underset{\sim}{W} = \text{Inv Lim}\,\{W_A, p_{AB}\}_{\mathcal{M}},$$

where W_A is—for a finite number of $A \in \mathcal{M}$—*one* component of $p_{AG}^{-1}(U)$, while $W_A = p_{AG}^{-1}(U)$ for the remaining $A \in \mathcal{M}$. Since X_G is a Hausdorff space, X_* has the same property, and the maps q_A in (3.2) are continuous. Also X_* is connected, since each X_A is connected. If for example each $A \in \mathcal{M}$ has finite index in G, then X_* will be locally compact, because in this case each W_A in (4.2) has compact closure in X_A (and an inverse limit of compact Hausdorff spaces is compact).

We shall now describe the relation between the sets $\underset{\sim}{W}$ in (4.5) and $\mathfrak{U}_x$ in (4.3). *We shall assume that $\mathcal{M}$ is linearly ordered;* then there is a smallest $D \in \mathcal{M}$ such that W_A in (4.5) is a single component of $p_{AG}^{-1}(U)$ if $A \supseteq D$, while W_A is not connected if $D \supset A$. Choose some set $\mathfrak{U}_x$ of the form (4.3) such that $U_A = W_A$ if $A \supseteq D$. The fact that $\mathcal{M}$ is linearly ordered also implies that $p_{BG}^{-1}(U) = W_B$ if $B \subseteq D$. Hence by (4.2),

(4.6) $$\underset{\sim}{W} = |D_\infty \cdot \mathfrak{U}_x| ,$$

where $D_\infty = \ker(G_\infty \to D)$.

The second topology we assign to X_∞ is that in which a basis of neighbourhoods at $x \in X_\infty$ is the family of all the special sets $\mathfrak{U}_x$ as U runs through all canonical neighbourhoods of x_G in X_G. Let the resulting topological space be denoted by S. Since $\mathfrak{U}_x$ in (4.6) is contained in $\underset{\sim}{W}$, the function

(4.7) $$\theta\colon S \to X_*$$

induced by the identity on X_∞ is *continuous*, while the functions $q_A\colon S \to X_A$ obtained from diagram (3.2) are local homeomorphisms, because $q_A \mid \mathfrak{U}_x$ is a homeomorphism onto $\mathfrak{U}_x$. We now recall from (3.5) the function $\phi\colon X_J \to X_\infty$, and we continue to denote by ϕ the induced function $X_J \to S$.

LEMMA 5. *$\phi\colon X_J \to S$ is continuous.*

Proof. If $y \in X_J$, then ϕ is continuous at y; for if $\mathfrak{U}_x$ is a neighbourhood of $x = \phi(y)$, then $\mathfrak{U}_x = \phi(V)$, where V is the component of $p_{JG}^{-1}(U)$ containing x, by definition of ϕ in (3.5) and of $\mathfrak{U}_x$ in (4.3). This establishes the lemma.

Since $\phi(o_J) = o_\infty$ and X_J is path-connected, ϕ maps X_J into the component T of S containing o_∞. The component T is open and path-connected, because each $\mathfrak{U}_x$ is connected. We now write $\phi\colon X_J \to T$.

LEMMA 6. *$\phi: X_J \to T$ is a homeomorphism.*

Proof. If $x \in T$, there is a path λ in T from o_∞ to x. Hence (see (3.2)), $q_G \circ \lambda$ is a path in X_G from o_G to $q_G(x)$ which lifts uniquely to a path μ in X_J from o_J to some $y \in p_{JG}^{-1}(q_G(x))$. Since paths in each X_A lift uniquely from X_G, while $q_A \circ \phi = p_{JA}$, it follows that $\phi \circ \mu = \lambda$, whence $x = \phi(y)$ and ϕ is onto. The continuity of ϕ^{-1} follows from the fact that p_{JG} and q_G are local homeomorphisms (observed after (4.7)) while $q_G^{\prime} \circ \phi = p_{JG}$. This completes the proof.

LEMMA 7. *G/J leaves T invariant* (that is, the orbit of $x \in T$ lies entirely in T).

Proof. For some $y \in X_J$, $x = \phi(y)$; therefore $gJ \cdot x = \phi(gJ \cdot y) \in T$, by (3.6).

Recalling (3.8), we easily obtain the following result.

LEMMA 8. *If $x \in T$, then $|G_\infty \cdot x| \cap T = |G/J \cdot x|$ and*

$$(q_G | T)^{-1} x_G = q_G^{-1}(x_G) \cap T.$$

5. HOMOLOGY GROUPS

We use Čech homology with compact carriers; that is to say, if Φ denotes the family of all compact subsets of a space Y, directed by inclusion, then we use the Direct Limit to define

$$H_1(Y) = \text{Dir Lim}\,\{H_1(F),\, i_{FF'}\}_\Phi, \tag{5.1}$$

where F and F' run through Φ, $i_{FF'}$ is induced by the inclusion $F \subseteq F'$, and $H_1(F)$ is the ordinary Čech group [7, Chapter V]. It is well known that if Y is LC^1 (for example), the singular homology group $H_1S(Y)$ is naturally isomorphic to $H_1(Y)$. The following is a consequence of Lemma 6.

LEMMA 9. *The map $\phi: X_J \to T$ induces an isomorphism $\phi_*: H_1(X_J) \to H_1(T)$.*

Our next result concerns the map $\theta: T \to X_*$, obtained by restricting $\theta: S \to X_*$ in (4.7) to the component T of S containing o_∞.

LEMMA 10. *The restriction $\theta: T \to X_*$ induces a monomorphism $\theta_*: H_1(T) \to H_1(X_*)$.*

Proof. By definition of Direct Limit, it suffices to show that if z is a Čech 1-cycle on a compact $F \subseteq T$ such that $\theta_* z$ bounds on a compact $K \subseteq X_*$, then z bounds also on K (for θ is induced by the identity on X_∞). We may obviously suppose $K \subseteq T$. Consider first the definition of θ_*. Let $\mathrm{Cov}_T\, K$ (respectively, $\mathrm{Cov}_*\, K$) denote the families of finite coverings of K in T (respectively, X_*) in the sense of (2.1). Then z is a Čech cycle $\{z(\mathscr{U})\}$, where each $z(\mathscr{U})$ is a 1-cycle on the nerve $N\mathscr{U}$ ($\mathscr{U} \in \mathrm{Cov}_T\, K$); also, $\theta_* z$ is a Čech cycle $w = w(\mathscr{V})$, where $\mathscr{V}$ runs through $\mathrm{Cov}_*\, K$ and $w(\mathscr{V}) = z(\theta^{-1}\mathscr{V})$. Since $\theta: T \to X_*$ is induced by the inclusion of T in X_*, we see that $\theta^{-1}\mathscr{V} = \{V \cap T \mid V \in \mathscr{V}\}$. Now, by (4.6), each $V \in \mathscr{V}$ can be written in the form $D_\infty^V \cdot \mathfrak{B}$, for some $D^V \in \mathscr{M}$ and some special set $\mathfrak{B} = \mathfrak{U}_x$ in T; hence, by Lemma 8,

(i) $V \cap T = |D^V/J \cdot \mathfrak{B}|$.

In particular, then, if $\mathscr{Q} \in \mathrm{Cov}_T\, K$ consists entirely of special sets $\mathfrak{U}_x$, then $\mathscr{V} = |G_\infty \cdot \mathscr{Q}|$ lies in $\mathrm{Cov}_* K$ and

(ii) $\theta^{-1}\mathscr{V} = |G/J \cdot \mathscr{Q}|$.

To prove that $z \sim 0$ on K, we must show that if $\mathscr{U} \in \mathrm{Cov}_T K$, then $z(\mathscr{U}) \sim 0$ on $N\mathscr{U}$. Let $\mathscr{Q} \in \mathrm{Cov}_T K$ refine $\mathscr{U}$ and satisfy the conditions on $\mathscr{W}$ in (2.4) relative to the group $\Gamma = G/J$; we may suppose also that each $Q \in \mathscr{Q}$ is of the form $\mathfrak{U}_{\dot{x}}$ in T. Then, by (ii), $w(\mathscr{V}) = z(\theta^{-1}\mathscr{V}) = z(|G/J \cdot \mathscr{Q}|)$ is a coordinate of $w = \theta_* z$; therefore $w(\mathscr{V}) \sim 0$ on $N(\mathscr{V})$ —that is, on $N(|G/J \cdot \mathscr{Q}|)$ by (i), since $K \subseteq T$. Now $w(\mathscr{V}) = f(z(\mathscr{Q}))$, where f is defined as in Lemma 3; whence, by that lemma, $z(\mathscr{Q}) \sim 0$ on $N\mathscr{Q}$. Since $\mathscr{Q}$ refines $\mathscr{U}$, we may project $z(\mathscr{Q})$ onto $N(\mathscr{U})$ by a projection π, to obtain $\pi z(\mathscr{Q}) \sim 0$. But $z(\mathscr{U}) \sim \pi z(\mathscr{Q})$, because z is a Čech cycle. This proves that $z \sim 0$ on K, as required, and it completes the proof.

Combining Lemmas 9 and 10, we obtain the main result as stated in the Introduction.

THEOREM 1. *If* $\mathscr{M}$ *is linearly ordered and* $J = \bigcap(M \mid M \in \mathscr{M})$, *then the map* $\theta \circ \phi\colon X_J \to X_*$ *induces a monomorphism* $H_1(X_J) \to H_1(X_*)$.

6. SOME APPLICATIONS

To indicate briefly three applications of the main theorem, we consider the residual properties of finiteness, solvability, and nilpotency. In each case, we may (by cofinality) take $\mathscr{M}$ to be a sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq G_{n+1} \supseteq \cdots, \tag{6.1}$$

where G_n is normal in G $(n = 0, 1, 2, \cdots)$; we now denote the associated covering maps by $p_n\colon X_{n+1} \to X_n$ with o_n as base-point of X_n.

(A) *Residual Finiteness.* For simplicity, we shall assume in this section that X is a compact LC^1 space. Then, by [2, Lemma 3.15], $G = \pi_1(X)$ is finitely generated; but more can be said, as we see in Lemma 11 below. For example, every finitely presented group is of the form $\pi_1(X)$, where X is a finite complex. We shall impose two further, "algebraic" conditions on G, and Theorem 2 below shows that then G is residually finite. We then prove Theorem 3 by verifying these conditions for the fundamental group of a compact surface. They might well be stated in a weaker form, but the choice made here has the virtue of directness.

Thus, we first suppose that G contains a sequence $\mathscr{M}$ of the form (6.1) such that

(6.2) *for each* $G_n \in \mathscr{M}$, *the index of* G_{n+1} *in* G_n *is finite and greater than* 1.

Denote by H^a the abelianizer of any group H, and let PH denote the periodic subgroup of H. Define FH to be H^a/PH; then for each n we have a short exact sequence

$$0 \to PG_n \to G_n^a \to FG_n \to 0.$$

Taking inverse limits, we obtain an exact sequence

$$0 \to \mathrm{Inv\ Lim}\ PG_n \to \mathrm{Inv\ Lim}\ G_n^a \to \mathrm{Inv\ Lim}\ FG_n, \tag{6.3}$$

where the homomorphisms induced by the inclusions $G_{n+1} \subseteq G_n$ are omitted from the Inv Lim notation.

LEMMA 11. $\mathrm{Inv\ Lim}\ G_n^a \approx H_1(X_*)$.

Proof. By (6.2), the covering space X_n of X, associated with G_n, is compact and LC^1 because X has those properties. Hence there is a natural isomorphism $H_1 S(X_n) \approx H_1(X_n)$ between the singular and Čech groups with integer coefficients. Also there is a natural isomorphism $G_n^a \approx H_1 S(X_n)$. We may therefore take Inverse Limits; now $X_* = \text{Inv Lim}(X_n, p_n)$ and the Čech functor H_1 is continuous (see [1, Chapter X]), whence

$$\text{Inv Lim } G_n^a \approx \text{Inv Lim}(H_1(X_n), p_{n*}) = H_1(X_*),$$

as required.

Next we suppose that G satisfies the following curious condition, which we shall discuss, together with an example (see (6.6)).

(6.4) *If* $A \subseteq C$ *are finitely generated subgroups of* G, A *is normal in* C, *and* $C/A \approx Z_p$ (for some prime number p), *then the induced homomorphism* $FA \rightarrow FC$ *is not onto.*

A stronger statement can now be made when G satisfies (6.4).

LEMMA 12. *Let* A *and* C *be as in* (6.4), *but suppose that* C/A *is finite and non-trivial. Then the homomorphism* $FA \rightarrow FC$ *is still not onto.*

Proof. By the Sylow Theorems, there is a subgroup U of C/A which is isomorphic to Z_p for some prime p. Hence there exists a subgroup B of C such that $A \subseteq B \subseteq C$ and $U = B/A$. These inclusions induce homomorphisms $FA \xrightarrow{\alpha} FB \xrightarrow{\beta} FC$, and the inclusion of A in C induces $\beta \circ \alpha$. Suppose, to get a contradiction, that $\beta \circ \alpha$ is onto; then β is onto. Now the groups FA, FB, FC are finitely generated, because the same is true of C; and the index of FB in FC is finite, not exceeding that of A in C. Moreover, FB and FC are free abelian. Hence a consideration of bases for FB and $\text{Im}(\alpha)$ shows that $FB = \text{Im}(\alpha)$, since the (false) hypothesis above implies that β maps both groups onto FC. Thus α would be onto, contrary to (6.4), when B/A replaces C/A. Hence $\beta \circ \alpha$ is not onto, and the Lemma is proved.

The following theorem will now be proved; we remind the reader that $G = \pi_1(X)$, where X is compact LC^1. It would be desirable to have an algebraic characterization of all such G which satisfy the conditions of the theorem.

THEOREM 2. *If* G *is finitely generated and satisfies* (6.4) *while* $\mathcal{M} = \{G_n\}$ *satisfies* (6.2), *then* $\text{Inv Lim } FG_n = 0$. *Moreover, the abelianizer* J^a *of* $J = \bigcap_n G_n$ *is isomorphic to a subgroup of* $\text{Inv Lim } PG_n$.

Proof. By Theorem 1, J^a is isomorphic to a subgroup of $H_1(X_*)$ and hence to a subgroup of $\text{Inv Lim } G_n^a$ by Lemma 11. Therefore, by exactness in (6.3), Theorem 2 will follow if we can prove that $\text{Inv Lim } FG_n = 0$.

As we observed above, G is finitely generated because X is compact and LC^1; therefore, by (6.2), G_n is finitely generated. Hence G_n^a and FG_n are also finitely generated. We have to show that if $\langle x_n \rangle \in \text{Inv Lim } FG_n$, then $x_n = 0 \in FG_n$, for each n. But for each n, if V_m denotes $\text{Im}(FG_m \rightarrow FG_n)$, then $x_n \in \bigcap_{m \geq n} V_m$. Now

$$V_n \supseteq V_{n+1} \supseteq \cdots \supseteq V_m \supseteq \cdots \quad \text{in } FG_n,$$

and for each m the index $|V_m : V_{m+1}|$ is finite, by (6.3), since it does not exceed $|G_m : G_{m+1}|$. Therefore, since FG_n is a finitely generated free abelian group, the intersection of the subgroups V_m will be zero, provided $|V_m : V_{m+1}| > 1$ for all

m. But this is ensured by (6.2) and Lemma 12 (with A and C taken to be G_{m+1} and G_m, respectively). Hence $x_n = 0$ for each n. Therefore Inv Lim $FG_n = 0$, and the proof is complete.

Denote by $\mathcal{N}$ the family of all normal subgroups of finite index in G, and let $\mathcal{N}$ be ordered by inclusion. Let $J(\mathcal{N})$ denote the intersection of all the subgroups in $\mathcal{N}$. Then we have at once a corollary of Theorem 2.

COROLLARY. *Suppose that* $\mathcal{M}$ *is cofinal in* $\mathcal{N}$, *and that* Inv Lim $PG_n = 0$ $(G_n \in \mathcal{M})$. *Then* $J(\mathcal{N})$ *lies in the perfect subgroup* $J = J(\mathcal{M}) \subseteq G$.

Consider now the case when G is finitely generated and of the form $\pi_1(Y)$, where Y is a (connected) surface. If Y has a boundary or is not compact, then G is free (see [3, 3.2]); but otherwise G is not free.

THEOREM 3. *If* X *is a surface and* $G = \pi_1(Y)$ *is finitely generated, then* G *is residually finite.*

Proof. If Y is not orientable, then it has an orientable double covering Y_o corresponding to a subgroup G_o of index 2 in G. Thus G is residually finite if and only if the same holds for G_o; it therefore suffices to consider the case when Y is orientable, and $Y \not\approx S^2$ (since $\pi_1(S^2)$ is trivial). Then $G = \pi_1(Y)$ is infinite, and finitely generated; therefore, by [2, Corollary 3.4], Y contains a compact subsurface X such that X is a deformation retract of Y, and $H_1(X) \approx H_1(Y)$. Hence G is of the form $\pi_1(X)$, where X is compact (metric) and LC^1.

If U is a subgroup of G, the corresponding covering space X_U is also an orientable surface with a free abelian group

$$H_1(X_U) \approx FU \approx U^a. \tag{6.5}$$

We now show that G satisfies condition (6.4). Since C/A is abelian, the commutator subgroups satisfy the condition $[A, A] \subseteq [C, C] \subseteq A$; therefore the homomorphism $f: FA \to FC$ factors as

$$FA \to A/[C, C] \xrightarrow{g} FC,$$

because $FA \approx A/[A, A]$, by (6.5). An isomorphism holds similarly for FC, whence $FC/Im(g) \approx C/A \neq \{1\}$. Therefore g, and hence f, is not onto. This verifies condition (6.4).

Next we find a sequence $\mathcal{M}$ which is cofinal in $\mathcal{N}$ and satisfies condition (6.2). Recall that G is finitely generated; then we may use the well-known result that, for each integer $r > 0$, there is only a finite number of subgroups of G of index r. Hence the intersection G_r of these is normal and of finite index in G. It also follows from (6.5) that the normal subgroup of G_r generated by all squares in G_r is of finite index greater than 1 in G_r. Therefore the sequence $\{G_r\}$ contains a subsequence $\mathcal{M}$ that satisfies (6.2) and is cofinal in $\mathcal{N}$, by construction.

The subgroup U in (6.5) cannot be perfect unless $U = \{1\}$; moreover, PU is always trivial. Hence, by the last corollary, $J(\mathcal{N}) \subseteq J(\mathcal{M}) = \{1\}$. Therefore G is residually finite, and Theorem 3 is proved.

More generally, suppose that G is a finitely generated Fuchsian group. It is well known that G contains a normal subgroup U, of finite index and without elements of finite order (U is then the group of a surface). Hence G is residually finite, since U is residually finite by Theorem 3. A direct proof can be given on the lines of that of Theorem 3, by representing G in the form $\pi_1(X)$, where X is a 2-dimensional cell-complex.

On the other hand, the 1-relator group with presentation

(6.6) $$Q = \langle a, b: ab^2a^{-1} = b^3 \rangle$$

is known to be non-Hopfian, and it is therefore not residually finite (see [5, pp. 260 and 413]). Hence one of the two conditions (6.2) and (6.4) does not hold in Q. These conditions are related, but we are as yet unable to clarify the situation.

(B) *Residual Solvability.* We now take the sequence $\mathcal{M} = \{G_n\}$ to be the derived series of G_o; that is, G_{n+1} is the commutator subgroup $[G_n, G_n]$ of G_n.

THEOREM 4. *If the sequence* $\{G_n\}$ *of derived groups of* G *does not terminate, then the abelianizer of their intersection is zero.*

Proof. We prove that $H_1(X_*)$ is zero, where X_* denotes the inverse limit of the system $\{X_n, p_n\}$ defined after (6.1). The family Ψ of compact subsets F of X_*, each of the form

(i) $$F = \text{Inv Lim}\{q_n F, p_{nm}\}_{n \geq m=0,1,2,\cdots},$$

is cofinal in the family of *all* compact subsets of X_* (directed by inclusion). Therefore, in the notation of (5.1),

(ii) $$H_1(X_*) \approx \text{Dir Lim}\{H_1(F), i_{FF'}\}_{F\epsilon\Psi}.$$

But, using the "continuity" property of the Čech functor [1, Chapter X] we see by the definition of F in (i) that

$$H_1(F) \approx \text{Inv Lim}\{H_1(q_n F), r_{nm}\},$$

where r_{nm} is induced by $p_{nm}: q_n F \to q_m F$ in (i). Also, by the diagram (3.2), $p_{nm}(q_n F) = q_m F$. We shall show that for each F there exists a compact $D \subseteq X_*$ such that $F \subseteq D$ and $H_1(F) \to H_1(D)$ is zero; this will prove that $H_1(X_*) = 0$.

We may suppose that F is path-connected, and we may choose $E_o \subseteq X_o$ as a compact connected neighbourhood of F_o. Then, by lifting canonical neighbourhoods, let $E_n \subseteq X_n$ be a compact connected neighbourhood of F_n such that $p_n(E_{n+1}) = E_n$ ($n = 0, 1, \cdots$). Choose a base-point $\langle y_n \rangle \in F$; then $y_n \in F_n$, and since each X_n is metric and LC^1, the image $\pi_1(F_n \mid E_n)$ of the homomorphism

$$\pi_1(F_n, y_n) \to \pi_1(E_n, y_n)$$

is finitely generated, by [2, Lemma 3.15]. Hence, for each $n > 0$, there exists a compact connected neighbourhood D_{n-1} of E_{n-1} such that for each of the finitely many generating loops g of $\pi_1(F_n \mid E_n)$, $p_n(g)$ is homotopic on D_n to a product of commutator loops; for $G_n = [G_{n-1}, G_{n-1}]$, while loops and their homotopies have compact carriers. Therefore the same holds when g is replaced by any y_n-based loop λ in F_n. Now, every singular 1-cycle z on F_n is homologous on F_n to a y_n-based loop λ; thus $p_n(z)$ is homologous on F_{n-1} to $p_n(\lambda)$, which in turn is homologous to zero on D_{n-1}. It follows that $H_1S(F_n) \to H_1S(D_{n-1})$ is the trivial homomorphism of singular homology groups; hence the analogous homomorphism with Čech homology is also trivial. As with the E's, we may assume that $p_n(D_{n+1}) \subseteq D_n$ ($n = 0, 1, 2, \cdots$), and then form $D = \text{Inv Lim}\{D_n, p_n\} \subseteq X_*$. Thus D is compact as an inverse limit of compact sets, and $F \subseteq D$. By construction (since the sequence $\{G_n\}$ does not terminate), the homomorphism $H_1(F) \to H_1(D)$ is trivial, and therefore, by the remark above, $H_1(X_*) = 0$. Hence, by Theorem 1, the abelianizer of J is trivial, and Theorem 4 is established.

Thus, Fuchsian groups (in particular, free groups and surface groups) are residually solvable, since they possess no perfect nontrivial subgroups.

(C) *Residual Nilpotency.* Let the sequence $\mathcal{M} = \{G_n\}$ be the lower central series of G; that is to say, let $G_o = G$, $G_{n+1} = [G_n, G]$. In general, it is difficult to calculate the homomorphisms $H_1(X_n) \to H_1(X_m)$ induced by the covering maps p_{nm}. In this section, we merely indicate a proof of the known result:

(6.7) *when* G *is free, the intersection* J *of all the* G_n *is trivial.*

If J is not trivial, there exists an essential o-based loop g in X, representing an element of J; to obtain a contradiction, it suffices by Theorem 1 to show that the image of g in each $H_1(X_m)$ is zero. We claim, for each $n > 0$, that if g lifts into X_n as an o_n-based loop g', then there are o_{n-1}-based loops g_i in X_{n-1} such that under the map $p_{n,n-1}: X_n \to X_{n-1}$,

(i) $$p_{n,n-1}(g') \sim \sum_i (T_i g_i - g_i)$$

for suitable covering transformations $T_i \in G/G_{n-1}$.

To see this, we observe that the projection $p_{n1}: X_n \to X_1 = X$ maps g' onto the o-based loop g; and since $J \subseteq G_n$, g is homotopic relative to o in X to a product of commutator loops of the form $c = [k, f]$, where k and f are loops representing elements of G_{n-1} and G, respectively. If we lift c into X_{n-1}, we obtain a loop of the form $c' = j^{-1} e(T_f j)e^{-1}$, where j and e are paths covering k and f in X_{n-1}, and where T_f is the covering transformation of X_{n-1} induced in G/G_{n-1} by f. Now k represents an element of G_{n-1}, and therefore j is a loop; thus

$$p_{n,n-1}(g') \sim \sum c' \sim \sum (T_f j - j) \quad \text{on } X_{n-1},$$

the sum over all the factors c composing the product g. This establishes (i).

In the case when G is free, we may suppose X to be a one-dimensional complex. Let $\ell(g')$ denote the number of edges of X_n appearing with nonzero weight in the 1-*chain* g' (integer coefficients). Then, by (i),

(ii) $$\ell(p_{n,n-1}(g')) < 2 \sum \ell(g_i) < \ell(g'),$$

since the edges in the "tails" e mentioned above are deleted unless e is zero, in which case $c' \sim 0$ anyway. Now, if $\ell(x) = 0$, then x represents a commutator and $x \sim 0$; thus, if $n = m \cdot 2^n$, then by (ii) $p_{nm}(g') \sim 0$ in X_m, so that the image of g in $H_1(X_m)$ is zero. This establishes (6.7).

This particular proof of (6.7) can not be generalised in any obvious way. One must keep in mind that for knot-groups G, we have $G_2 = G_3 = \cdots$ (see [6, p. 59]), and that the trefoil knot-group is a 1-relator group.

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