NEIGHBORHOODS OF SURFACES IN 3-MANIFOLDS

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Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

Let S be a closed (that is, compact and boundaryless) 2-manifold topologically embedded in a two-sided manner in Int M, where M is a piecewise linear 3-manifold. The main result in this paper (Theorem 2) is that, arbitrarily close to S, there exists a polyhedral neighborhood of S, homeomorphic to $S \times [0, 1]$ with finitely many "small" handles of index 1 attached. In particular, if S is orientable, some neighborhood of S is embeddable in Euclidean 3-dimensional space E^3. In this sense, we can study many pathological embeddings in 3-manifolds without leaving E^3.

These results continue the line of investigation begun in [15] (see [16] for a survey of the results to be found in both papers), and we rely on some of that work, as well as on many of R. H. Bing's theorems (references [2] to [9]). We are also indebted to Professor Bing for many helpful discussions on these topics.

Using the above notation, and assuming that M - S has components U_0 and U_1, we say that S is *locally tame from* U_0 *at* $p \in S$ if the closure of U_0 is a topological 3-manifold at p. If the closure of U_0 is a 3-manifold, we say that S is *tame from* U_0. The term "manifold" will always refer to a *connected* set. When we wish to emphasize that a manifold possesses a combinatorial triangulation, we shall use the prefix "piecewise linear" (abbreviated: pwl), even though each topological manifold of dimension 3 or less is known to be a piecewise linear manifold. By a *cube-with-handles*, we mean a 3-manifold homeomorphic to the regular neighborhood in E^3 of a finite, connected graph. In considering a mapping $f: X \times [0, 1] \rightarrow Y$, we shall sometimes use the notation $f_t: X \rightarrow Y$ ($t \in [0, 1]$) to mean the mapping defined by $f_t(x) = f(x, t)$. Similar notation will refer to an f with domain $X \times [-1, 1]$.

By a *null-sequence* $E_1, E_2, \cdots$ of subsets of a metric space we mean a sequence such that the diameters of its elements converge to zero. Let S be a closed 2-manifold topologically embedded in Int M, where M is a piecewise linear 3-manifold. Let $X \subset S$ be a closed set, and let $U_1, U_2, \cdots$ be the components of S - X. We shall call X an S-*curve* if $\overline{U}_1, \overline{U}_2, \cdots$ is a null-sequence of mutually exclusive 2-cells with $\bigcup_i U_i$ dense in S. In case S is a 2-sphere, such an X is called a *Sierpinski curve* (see [5, Section 3]). We call

$$S - \bigcup_i \overline{U}_i \subset X$$

the *inaccessible part* of X. We shall say that an S-curve X is *tame* in M if for each 2-manifold J that is homeomorphic to S, contains X, and is locally tame at each point of J - X, it follows that J is tame in M. If S is a 2-sphere, then a

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Sierpinski curve $X \subset S$ is tame by this definition if and only if it is tame by Bing's definition (see [8], for example).

2. SURFACES TAME FROM ONE SIDE

Theorem 1 below implies Theorem 2 in case the 2-manifold under consideration is tame from one of its complementary domains (although Theorem 1 says more than that). We first state a result from [15].

LEMMA 0. *Let* B *be a* q*-cell* (q = 1, 2, *or* 3) *topologically embedded in* Int M, *where* M *is a piecewise linear* 3*-manifold, and let* $D \subset \partial B$ *be a* (q - 1)*-cell. Let* $A_1, A_2, \cdots, A_k$ *be a finite disjoint collection of tame arcs in* M - D *with* $\partial A_i \subset M - B$ *for each* i. *Then there exists a compact set* $C \subset B - D$ *such that, for each* $\varepsilon > 0$, *there is a piecewise linear homeomorphism* h: M → M, *equal to the identity outside the* ε*-neighborhood of* C, *and with each* $h(A_i)$ *contained in* M - B.

Except for the assertion that h can be chosen to be pwl (which follows from [2; Theorem 3]), Lemma 0 is simply Corollary 1.2 of [15].

LEMMA 1. *Let* S *be a closed, piecewise linear* 2*-manifold. Then, for each* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that if each of* $D_1, D_2, \cdots, D_p$ *is a finite, disjoint collection of closed* 2*-cells in* S *with the elements of* D_i *disjoint from those of* D_j *for* $i \neq j$ *and with the union of the elements of each* D_i *having diameter less than* δ, *then there exist disjoint, closed* 2*-cells* $F_1, F_2, \cdots, F_p$ *in* S, *each having diameter less than* ε, *and such that all the elements of* D_i *are contained in the interior of* F_i (i = 1, 2, ⋯, p).

Proof. We shall assume that S is a geometric complex, rectilinearly embedded, with respect to a triangulation T^*, in some Euclidean space. Since there exists a uniformly continuous homeomorphism between any two metrized copies of S, the conclusion will follow in general.

Let T (a subdivision of T^*) be a triangulation of S such that the star of each vertex of T has diameter less than ε/7, and let T' and T" denote the first and second barycentric subdivisions of T, respectively. If $s \in T$, we shall denote the barycenter (in T') of s by b(s), the closed star of b(s) in T" by B(s), and the open star of B(s) in T" by OB(s). Note that B(s) is a closed 2-cell, OB(s) is an open 2-cell containing B(s), and $\{B(s) \mid s \in T\}$ is a covering of S. We choose $\delta > 0$ to be less than half the minimum distance from any set B(s) to the complement of OB(s) in S.

Now let $D_1, D_2, \cdots, D_p$ have the properties stated in the lemma, and let

$$A = \{D \mid D \in D_i,\ 1 \leq i \leq p\}$$

(this is a disjoint collection of 2-cells). Let us say that D_k is of *type* j (j = 0, 1, or 2) if j is the largest integer for which there exists a j-simplex $s \in T$ with B(s) meeting an element of D_k. If D_k is of type j, then each element of D_k is also said to be of type j. Clearly, if D_k is of type j, then $\bigcup\{D \mid D \in D_k\} \subset OB(s)$, for some unique j-simplex s.

Now if s and t are distinct j-simplexes of T, then $OB(s) \cap OB(t) = \square$. Further, if $s \in T$ is a j-simplex, and if each of E and F is an element of A each of whose points is nearer to B(s) than to S - OB(s), then some arc from E to F in OB(s) meets no elements of A other than E and F. It follows that for j = 0, 1, 2, there exists a disjoint collection of closed 2-cells $H_1^j, H_2^j, \cdots, H_{m(j)}^j$ (where $m(j)$ is the number of 2-cells D_k of type j, and where $m(0) + m(1) + m(2) = p$) with the following three properties: no H_i^j intersects an element of A of type different from j; each H_i^j lies in OB(s) for some j-simplex s (and hence H_i^j has diameter less than $\varepsilon/7$); and each H_i^j contains all the elements of exactly one D_k of type j. Of course, it may happen that $H_i^j \cap H_h^k \neq \Box$ for $j \neq k$.

Define $F_i = H_i^2$ for $i \leq m(2)$. Since each of these F_i contains a point not in any H_j^1 (in fact, a point in an element of A of type 2), there exists a homeomorphism $f_1: S \to S$, equal to the identity outside a small neighborhood of $\bigcup_{i=1}^{m(2)} F_i$, and such that

$$f_1(H_j^1) \cap \bigcup_{i=1}^{m(2)} F_i = \Box$$

for each j. This neighborhood meets no element of A of type 0 or 1, and each component of the neighborhood has diameter less than $\varepsilon/7$. Now put

$$F_i = f_1(H_i^1) \quad \text{for } m(2) < i \leq m(1) + m(2).$$

Note that each of these new F_i has diameter less than $3\varepsilon/7$ and that each F_i thus far defined contains all the elements of exactly one D_k of type 1 or 2.

We use the same procedure to obtain the rest of the F_i. That is, there exists a homeomorphism $f_0: S \to S$, equal to the identity outside a small neighborhood of $\bigcup_{i=1}^{m(1)+m(2)} F_i$, such that

$$f_0(H_j^0) \cap \bigcup_{i=1}^{m(1)+m(2)} F_i = \Box$$

for each j. This neighborhood meets no element of A of type 0, and each of its components has diameter less than $3\varepsilon/7$. Finally, put

$$F_i = f_0(H_i^0) \quad \text{for } m(1) + m(2) < i \leq p.$$

Note that each of these new F_i has diameter less than ε and that each of the F_i we have defined contains all the elements of exactly one D_k. This completes the proof.

The following is a modification of a result of Bing [8, Theorem 1.1].

LEMMA 2. *Let* S *be a closed, piecewise linear* 2*-manifold,* M *a piecewise linear* 3*-manifold, and* $h: S \to \text{Int } M$ *a homeomorphism such that* $M - h(S)$ *has components* U_{-1} *and* U_1*, with* $h(S)$ *tame from the* U_{-1}*-side. Then, for each positive number* ε*, some piecewise linear homeomorphism* $H: S \times [-1, 1] \to M$ *has the following four properties:*

(1) *for each* $x \in S$ *and each* $t \in [-1, 1]$ *the distance from* $h(x)$ *to* $H(x, t)$ *is less than* ε;

(2) $H_{-1}(S) \subset U_{-1}$;

(3) $\overline{U}_{-1} \cap H_1(S)$ *is covered by the interiors of a finite disjoint collection of* 2*-cells in* $H_1(S)$*, each of diameter less than* ε;

(4) *there exists a finite disjoint collection of topological* 3-*cells* $C_1, \cdots, C_k$ *in* $\overline{U}_{-1}$ *such that each* C_i *has diameter less than* ε *and meets* h(S) *precisely in a* 2-*cell, such that* $h(S) - H(S \times (-1, 1))$ *is covered by the interiors of these* 2-*cells, and such that* $(\partial C_i) - \operatorname{Int}(C_i \cap h(S)) \subset H(S \times (-1, 1))$.

Proof. Let δ be a positive number such that, for each homeomorphism $g: S \to M$ differing from h by less than δ, and for each compact set Z in g(S) whose components all have diameter less than δ, the image of g contains a finite disjoint collection of 2-cells of diameter less than ε whose interiors cover Z (see, for example, the proof of [6, Theorem 12]). By [8, Theorem 6.1] there exists a tame h(S)-curve such that each component of h(S) - X has diameter less than δ. Since h(S) is tame from the U_{-1}-side, we may deform $h|S - X$ slightly into U_{-1}, to obtain a homeomorphism $g: S \to \overline{U}_{-1}$, differing from h by less than δ, such that $g(S) \cap h(S) = X$, the 2-manifold g(S) is locally tame at each point of g(S) - X, and the closures of the components of $\overline{U}_{-1} - g(S)$ that meet h(S) form a null sequence of 3-cells $C_1, C_2, \cdots$, each of diameter less than δ.

Now, g(S) is tame, and since it is 2-sided, there exists a homeomorphism $G: S \times [-1, 1] \to M$, with $G_0 = g$, and having properties (1) and (2) of the lemma (restated for G). Since the C_i form a null-sequence, there exists an integer k such that $C_i \subset G(S \times (-1, 1))$ for $i > k$. Clearly, G has property (4) (restated for G), with respect to $C_1, \cdots, C_k$. By our choice of δ, G has property (3) (restated for G). Finally, we replace G by a sufficiently close pwl approximation [2, Theorem 2'] to obtain the required H.

THEOREM 1. *Let* M *be a piecewise linear* 3-*manifold, and* K *a compact piecewise linear* 3-*manifold with nonempty boundary, topologically embedded in* Int M. *Then, for each* $\varepsilon > 0$, *there exists a polyhedral subset* L *of* M *in the* ε-*neighborhood of* K, *homeomorphic to* K, *with* ∂L *homeomorphically within* ε *of* ∂K, *and such that there exists a finite disjoint collection* $H_1, H_2, \cdots, H_p$ *of polyhedral cubes-with-handles in* M, *each* H_i *having diameter less than* ε, *each* H_i *meeting* L *precisely in a* 2-*cell, and with*

$$K \subset \operatorname{Int}[L \cup H_1 \cup H_2 \cup \cdots \cup H_p].$$

Proof. Let S be a finite complex and h a homeomorphism of S onto ∂K. Let N be a product neighborhood of ∂K in K, and let $U_{-1} = \operatorname{Int} K$, $U_1 = M - K$. We choose a $\delta > 0$ to satisfy the following two conditions.

(1) If g is any homeomorphism of S into M differing from h by less than δ, and if $g(S) \subset U_{-1}$, then g(S) separates the closure of K - N from the closure of U_1.

(2) If g is any homeomorphism of S into M differing from h by less than δ, and if each of $D_1, \cdots, D_p$ is a finite, disjoint collection of closed 2-cells in g(S) with the elements of D_i disjoint from those of D_j for $i \neq j$ and with the union of the elements of each D_i having diameter less than δ, then there are disjoint closed 2-cells $F_1, \cdots, F_p$ in g(S), each of diameter less than $\varepsilon/2$, and such that all the elements of D_i are contained in the interior of F_i, for each i. In particular, $\delta < \varepsilon/2$.

(That condition (2) can be met for some $\delta > 0$ follows easily from Lemma 1.)

Since each component of h(S) is 2-sided, repeated applications of Lemma 2 give a pwl homeomorphism $H: S \times [-1, 1] \to M$ satisfying conditions (1) to (4) of that lemma for the positive number $\delta/3$. By our choice of δ, $H_{-1}(S)$ separates the closure of K - N from the closure of U_1. By [10; Theorem 1], the closure of each component of $N - H_{-1}(S)$ is homeomorphic to $S \times [0, 1]$. Hence, if we let L_0 be the closure of the component of $K - H_{-1}(S)$ not meeting ∂K, and if we write

$$L_1 = L_0 \cup H(S \times [-1, 1]),$$

then some homeomorphism of K onto L_1 is the identity on L_0. Note also that the only points of K not in Int L_1 are contained in the 3-cells $C_1, \cdots, C_k$ (provided by Lemma 2). We assume that no C_j lies in Int L_1.

Let $R_1, \cdots, R_k$ be a disjoint collection of compact, polyhedral, orientable 3-manifolds with boundary, each R_j having diameter less than $\delta/3$, with $C_j \subset \text{Int } R_j$, and with ∂R_j in general position with respect to $\partial L_1 = H_1(S)$. We also suppose, by condition (3) in Lemma 2, that each R_j is so close to C_j that $H_1(S) \cap \bigcup R_j$ is covered by the interiors of a finite, disjoint collection of 2-cells in $H_1(S)$, each of diameter less than $\delta/3$. Let $T_1, \cdots, T_p$ be the closures of the components of $\left[\bigcup R_j\right] - L_1$. Each T_i is a compact, polyhedral, orientable 3-manifold with boundary, and each component of $T_i \cap L_1 = (\partial T_i) \cap (\partial L_1)$ is a punctured 2-cell.

For each i, let a disjoint collection Z_i of polygonal arcs in T_i be selected such that for each $A \in Z_i$, (i) either

$$A \subset L_1 \quad \text{with } \partial A \subset \partial (T_i \cap L_1)$$

or

$$A \cap L_1 = \square \quad \text{with} \quad A \cap \partial T_i = \partial A,$$

and (ii) T_i minus a thin tubular neighborhood in T_i of each $A \in Z_i$ is a cube-with-handles Y_i meeting L_1 in a finite disjoint collection of polyhedral 2-cells in $(\partial Y_i) \cap (\partial L_1)$. The existence of such arcs follows from [15, Lemma 1] and the fact that each component of $T_i \cap L_1$ is a punctured 2-cell.

Now apply Lemma 0 once for each C_j, and piece together the resulting homeomorphisms. That is, in Lemma 0, for fixed j, take

$$q = 3, \quad B = C_j, \quad D = \partial C_j - \text{Int}(C_j \cap \partial K), \quad M = L_1 \cup \bigcup T_i,$$

and take the ε of Lemma 0 so small that the corresponding ε-neighborhoods of the compact subsets of the C_j have diameter less than $\delta/3$, and are disjoint from each other, from $\partial(L_1 \cup \bigcup T_i)$, and from the closure of $K - \bigcup C_j$. This gives a pwl homeomorphism G of $L_1 \cup \bigcup T_i$ onto itself that is the identity on $\partial(L_1 \cup \bigcup T_i)$, moves each point less than $\delta/3$, and satisfies the condition $G(A) \cap K = \square$ for each A in each Z_i.

Note that the diameter of each $G(T_i)$ is less than $\delta < \varepsilon/2$, and that GH_1 differs from h by less than $2\delta/3$. Let $L = G(L_1)$, and let W_i be $G(T_i)$ minus a thin, nice neighborhood in $G(T_i)$ of each $G(A)$ (A in some Z_i). Then each W_i is a polyhedral cube-with-handles, each component of $W_i \cap L$ is a 2-cell in the common boundary of W_i and L, and $K \subset \text{Int}[L \cup W_1 \cup \cdots \cup W_p]$.

Since each set $W_i \cap L$ has diameter less than δ and lies in $GH_1(S) = \partial L$, requirement (2) on δ implies that there exist disjoint polyhedral closed 2-cells $F_1, \cdots, F_p$ in ∂L, each of diameter less than $\varepsilon/2$, such that $W_i \cap L \subset \text{Int } F_i$ for each i. The required cube-with-handles H_i is then W_i plus a polyhedral 3-cell obtained by thickening F_i slightly in L.

3. THE GENERAL CASE

LEMMA 3. *Let* M *be a piecewise linear* 3*-manifold, and let* S *be a closed, piecewise linear* 2*-manifold topologically embedded in* Int M, *and separating* M. *Suppose that* S *is a retract of* M. *Then, for each* $\varepsilon > 0$ *there exist an open set* $U \subset M$ *and a* $\delta > 0$ *such that* $S \subset U$, *and such that if* S_0 *is any tame* 2*-manifold homeomorphically within* δ *of* S, *then* S_0 *separates* M, *and some retraction of* M *onto* S_0 *moves each point of* U *less than* ε.

Proof. Let $r: M \to S$ be a retraction. First, we require $\rho > 0$ to be less than $\varepsilon/2$ and so small that each mapping of S into M that moves each point less than ρ is homotopic in M to the inclusion $S \to M$. This ensures that each 2-manifold S_0 homeomorphically within ρ of S separates M. We also require that ρ be so small that if S_0 is homeomorphically within 2ρ of S, then each mapping of S_0 into itself that moves each point less than 2ρ is $\varepsilon/4$-homotopic (in S_0) to the identity. In other words, we require that for some homotopy $H_t: S_0 \to S_0$ (in which H_0 is the identity and H_1 coincides with the mapping in question), each path

$$\{H_t(x) \mid t \in [0, 1]\}$$

has diameter less than $\varepsilon/4$. Choose U so that $S \subset U$ and so that r moves each point of U less than $\rho/4$. Finally, take a positive δ less than ρ and less than the distance from S to M - U.

Now suppose that S_0 is tame and that some homeomorphism h of S_0 onto S moves each point less than δ. Let G be one of the components of $M - S_0$. It suffices to demonstrate a retraction r_0 of $\overline{G}$ onto S_0 that moves each point of $U \cap G$ less than ε.

Let N be a product neighborhood of S_0 in $U \cap \overline{G}$ (that is, let there exist a homeomorphism $g: S_0 \times [0, 1] \to N \subset U \cap \overline{G}$ such that $g(x, 0) = x$, for each $x \in S_0$) with each arc $\{g(x, t) \mid t \in [0, 1]\}$ of diameter less than $\delta/4$. Let $S_i = g(S_0 \times \{i\})$, and let $p_i: N \to S_i$ be the "projection" defined by $p_i\, g(x, t) = g(x, i)$ for $i = 0, 1$.

Note that S_1 is homeomorphically within $5\delta/4$ of S and that under $p_1 h^{-1} r: S_1 \to S_1$, each point of S_1 moves less than $3\rho/2$. This implies that some $\varepsilon/4$-homotopy on S_1 with the properties described in the first paragraph of this proof satisfies the condition $H_1 = p_1 h^{-1} r$. Define r_0 by the rule

$$r_0(y) = \begin{cases} h^{-1} r(y) & \text{if } y \in G - N, \\ p_0 H_t p_1(y) & \text{if } y = g(x, t) \in N. \end{cases}$$

We see that r_0 moves points of $(U \cap G) - N$ less than $5\rho/4 < 5\varepsilon/8$, and points of N less than $\varepsilon/2$. This completes the proof.

THEOREM 2. *Let* M *be a piecewise linear* 3*-manifold, and let* S *be a closed piecewise linear* 2*-manifold topologically embedded in* Int M, *and separating* M, *and let* $\varepsilon > 0$. *Then, for some polyhedral subset* L *of* M *in the* ε*-neighborhood of* S *such that* L *is homeomorphic to* $S \times [0, 1]$ *and such that each component of* ∂L *is homeomorphically within* ε *of* S, *there is a finite disjoint collection* $H_1, H_2, \cdots, H_p$ *of polyhedral cubes-with-handles in* M, *such that each* H_i *has diameter less than* ε, *each* H_i *meets* L *precisely in a* 2*-cell in* $(\partial H_i) \cap (\partial L)$, *and*

$$S \subset \text{Int}[L \cup H_1 \cup H_2 \cup \cdots \cup H_p].$$

Proof. Let M - S have components U_0 and U_1, and let M* be the subspace of $M \times [0, 1]$ consisting of all (x, t) for which

$$x \in U_0 \text{ and } t = 0, \quad \text{or } x \in S \text{ and } t \in [0, 1], \quad \text{or } x \in U_1 \text{ and } t = 1.$$

It follows from [3, Theorem 5] and either [11] or [13] that M* is a topological 3-manifold and hence has a piecewise linear triangulation $T^{\#}$. Let f be the mapping of M* onto M given by projection onto the M-coordinate.

If M has metric d, we assign to M* the metric d*, where

$$d^*[(x_1, t_1), (x_2, t_2)] = \sqrt{d(x_1, x_2)^2 + (t_1 - t_2)^2}.$$

We note for later use that if X is any subset of M* with finite diameter, then f(X) also has finite diameter, and diam f(X) $\leq$ diam X. Note that $f \mid \overline{U}_i \times \{i\}$ is an isometry onto $\overline{U}_i$, for i = 0, 1, and that for $x \in S$, $f^{-1}(x)$ is an arc. By [15, Corollary 1.3], some neighborhood in M* of each $f^{-1}(x)$ is embeddable in E^3, and hence, as in [14; Lemma 6], each set $f^{-1}(x)$ is cellular in M*.

Now let $\varepsilon > 0$. Let A and B be compact polyhedral 3-manifolds with nonempty boundary such that

$$S \subset \text{Int } B \subset B \subset \text{Int } A \subset A \subset M,$$

A retracts onto S, and there is a strong deformation retraction (in A) of B onto S. Let $\delta > 0$ be such that the δ-neighborhood of S in M lies in B and each subset of B of diameter less than δ lies in a closed polyhedral 3-cell of diameter less than $\varepsilon/2$ in B. Let $\mu > 0$ be such that each subset of S of diameter less than μ lies in an open 2-cell in S of diameter less than $\delta/8$.

The space M* contains subsets $L^*, H_1^*, H_2^*, \cdots, H_p^*$, that are polyhedral with respect to $T^{\#}$ and satisfying the conclusion of Theorem 1, with respect to the 3-manifold $K^* = S \times [0, 1] \subset M^*$ and the positive number μ. Let $R^* = L^* \cup H_1^* \cup \cdots \cup H_p^*$, so that $K^* \subset \text{Int } R^*$. We assume that μ is small enough so that we may speak meaningfully of a unique component of ∂L^* (or ∂R^*) as being *associated* with each component of ∂K^*.

By our previous assertions about the sets $f^{-1}(x)$ and by [1, Corollary 1], $R = f(R^*)$ is homeomorphic to R^*; by [2; Theorem 9], ∂R (and hence R) is tame. By applying a theorem of Moise (quoted in [2] as Theorem 2) to R and using [12, Corollary 3] to alter $T^{\#}$ slightly near ∂R^*, we may assume that a neighborhood of ∂R^* is polyhedral under a triangulation T^* of M* and that $\partial R = f(\partial R^*)$ is polyhedral relative to the given triangulation of M.

Note that R lies in the μ-neighborhood ($\mu < \delta/8 < \varepsilon/16$) of S and contains S in its interior, and that each set $f(H_i^*)$ has diameter less than μ. Further, since ∂L^* is homeomorphically within μ of ∂K^*, there exists, for each component C^* of ∂L^*, a mapping of $S = f(K^*)$ onto $f(C^*)$ that has a well-defined inverse on $f(C^*) \cap \partial R$ and moves each point of S less than μ.

We claim that each polyhedral, closed 2-manifold Z in R of diameter less than δ bounds a 3-manifold in R, of diameter less than $\varepsilon/2$, such that this 3-manifold can be piecewise linearly embedded in E^3. Indeed, it certainly bounds such a 3-manifold in B. Moreover, the 3-manifold is contractible to a point in B. If the 3-manifold does not lie in R, then Z must separate in R the two components of ∂R. Hence, some closed, polyhedral 2-manifold $W \subset \partial R$, considered as a 2-cycle (with Z_2 coefficients), bounds in A but not in R (in fact, it is the generator of $H_2(R)$). This is a contradiction, since the image of $H_2(R)$ in $H_2(A)$ under the inclusion homomorphism is the same as the image of $H_2(S)$ in $H_2(A)$ under the inclusion homomorphism, and the latter homomorphism has trivial kernel.

Consider the 2-cell $D_i^* = H_i^* \cap L^*$. Our main task is to show that each of the polyhedral simple closed curves $f(\partial D_i^*)$ bounds a polyhedral 2-cell $D_i \subset R$, of diameter less than $\delta/2$, such that

$$D_i \cap \partial R = \partial D_i \text{ for each } i \quad \text{and} \quad D_i \cap D_j = \square \text{ for } i \neq j .$$

Let ρ be a positive number, small enough so that

$$[\max_i \operatorname{diam} f(H_i^*)] + 2\rho < \mu ,$$

less than one-fourth of the least of the distances between 2-cells D_i^* and D_j^* $(i \neq j)$, and less than the distance between K^* and ∂R^*.

In the paragraphs to follow, the D_i will be constructed for those values of i for which D_i^* belongs to the component of ∂L^* associated with $S^* = S \times \{0\} \subset \partial K^*$. The phrase "for each i" should be interpreted accordingly, until this part of the proof is completed. The construction of the remaining D_i will be symmetric.

According to [8, Theorem 6.1], there exists a tame S-curve X such that if $E_1, E_2, \cdots$ are the disjoint 2-cells that are the closures of the components of S - X, then diam $S^* \cap f^{-1}(E_j) < \rho$. Let $E_j^* = S^* \cap f^{-1}(E_j)$. We also note from the proof of Theorem 6.1 of [8] that we can choose a tame 2-manifold J that contains X and is homeomorphically as close to S as we wish.

We claim that $X^* = S^* \cap f^{-1}(X)$ also lies in a tame 2-manifold $J^* \subset M^*$ that can be chosen homeomorphically as close to S^* as desired. To obtain J^*, we simply replace each E_j^* by a 2-cell that has the same boundary, is locally tame at each of its interior points, has its interior in Int K^*, and is homeomorphically close to E_j^*. If this construction is performed nicely, then J^* is clearly tame from one component of $M^* - J^*$, each arc $f^{-1}(x)$ $(x \in S)$ meets J^* in exactly one point, and J^* is locally tame from the other component at each point of $J^* - X^*$. It will also be convenient later if the closures of the components of $K^* - J^*$ that meet S^* form a null-sequence of topological 3-cells of diameter less than ρ. We leave to the reader the details of the proof (it uses the fact that $f(X^*) = X$ is tame) that the latter component of $M^* - J^*$ is locally simply connected at each point of X^* (see [7, Theorem 8.1]). This implies [3, Theorem 6] that J^* is tame in M^*.

By the methods of Bing [9, Theorem 1], we can deduce from these facts that there exists a homeomorphism $h: M^* \to M^*$ such that, for each i, $h(D_i^*) \cap X^*$ is the union of a finite number of mutually exclusive simple closed curves, each in the inaccessible part of X^*, such that X^* locally lies on different sides of $h(D_i^*)$ near these curves, h is the identity outside the ρ-neighborhood of $S^* \cap \bigcup D_i^*$, and h moves each point less than ρ. Further, Bing's proof shows that h can be chosen so that $h(D_i^*) \cap J^*$ lies in the inaccessible part of X^*, for each i, where J^* is the tame 2-manifold described in the preceding paragraph and is homeomorphically within ρ of S^*. Note that, for each i, $S^* \cap h(D_i^*)$ consists of the above simple closed curves plus the null sequence $\{h(D_i^*) \cap \text{Int } E_j^* \mid j = 1, 2, \cdots\}$ of *compact* sets of diameter less than ρ. Further, for $i \neq k$, the sets $h(D_i^*)$ and $h(D_k^*)$ are disjoint and no 2-cell E_j^* meets both $h(D_i^*)$ and $h(D_k^*)$. Also, $\partial h(D_i^*) = \partial D_i^*$, and

$$\text{diam } fh(D_k^*) \leq [\max_i \text{ diam } f(H_i^*)] + 2\rho < \mu .$$

We call a simple closed curve in $(h \mid D_i^*)^{-1}(X^*)$ a *maximal curve* if no other simple closed curve in $(h \mid D_i^*)^{-1}(X^*)$ separates it from ∂D_i^*. A *maximal 2-cell* in D_i^* is the 2-cell in D_i^* bounded by a maximal curve. Clearly the maximal 2-cells of D_i^* are disjoint, and we let V_i^* denote the punctured 2-cell obtained by removing their interiors from D_i^*.

Let F^* be the closure of the component of $R^* - S^*$ containing the component of ∂R^* associated with S^*. Recalling our description of J^* and the fact that $h(D_i^*) \cap J^* \subset X^*$, we see that $fh(V_i^*) \subset f(F^*)$ and that $V_i^* \cap (fh)^{-1}(X)$ consists precisely of the maximal curves in D_i^*. Further, the fh-images of V_i^* and V_j^* are disjoint, for $i \neq j$.

There exist an open set $U \subset \text{Int } R$ containing S and a $\delta_0 > 0$ that together satisfy the conclusion of Lemma 3 with respect to S, the 3-manifold R, and the positive number which is the minimum of $\delta/16$ and one-half the least of the distances between $fh(V_i^*)$ and $fh(V_j^*)$ $(i \neq j)$. As we remarked earlier, some tame 2-manifold J contains X, is homeomorphically within δ_0 of S, and is so close to S that the fh-image of each maximal curve of each D_i^* bounds a 2-cell in J of diameter less than $\delta/8$.

By Lemma 3, some retraction $r: R \to J$ moves each point of U less than $\delta/16$. Let r_0 be the retraction of R onto R_0 induced by r, where R_0 is the closure of the component of $R - J$ containing the component of ∂R under consideration. Define $g_i: V_i^* \to R_0$ by $g_i(x) = r_0 fh(x)$ $(x \in V_i^*)$. We remark that $g_i = fh$ on the maximal curves of D_i^*, and hence the g_i-image of each maximal curve in D_i^* bounds a 2-cell in J of diameter less than $\delta/8$. Further, g_i differs from $fh \mid V_i^*$ by less than $\delta/16$, and $g_i(V_i^*) \cap g_j(V_j^*) = \square$ for $i \neq j$. Hence, diam $g_i(V_i^*) < \mu + \delta/8 < \delta/4$.

Since J is tame and hence has a product neighborhood, we may assume that $g_i^{-1}(J)$ consists precisely of the maximal curves of D_i^*. We now obtain mutually exclusive singular 2-cells in R, of diameter less than $\delta/2$ and with the same "boundary" as $f(\partial D_i^*)$, by successively attaching to the g_i-image of each maximal curve the above 2-cell in J having the same boundary and diameter less than $\delta/8$, and then deforming this 2-cell slightly to one side of J. Of course, we must begin this process with the "innermost" (on S) images under the g_i's of the maximal curves of the D_i^*'s and work "outward," always deforming the next 2-cell to a lower "J-level" of the product neighborhood of J. The resulting singular 2-cells were also chosen so that they have diameter less than $\delta/4 + \delta/4 = \delta/2$. Finally, by [17], we can choose the desired nonsingular D_i to lie very close to the above singular 2-cells. As remarked before, the construction of the D_i having boundaries belonging to the other component of $R - S$ is symmetric, and we assume that it also has been completed.

For each i, the closed 2-manifold

$$Z_i = D_i \cup f[(\partial H_i^*) - \text{Int } D_i^*]$$

has diameter less than δ, and hence bounds a polyhedral 3-manifold H_i in R of diameter less than $\varepsilon/2$ such that H_i can be piecewise linearly embedded in E^3. We leave to the reader the proof that H_i is actually a cube-with-handles (note that H_i is a retract of R, so that the inclusion $H_i \to R$ induces a monomorphism on fundamental groups), and that if we define L to be the closure of R - $\bigcup H_i$, then L is homeomorphic to L*.

Finally, to verify that each component C of ∂ L is homeomorphically within ε of S, we remark that we can obtain C by removing certain singular 2-cells of diameter less than μ from f(C*) (where C* is a component of ∂ L* homeomorphically within μ of, say, S*) and replacing them with 2-cells of diameter less than $\delta/2$. Hence C is homeomorphically within $\mu + \mu + \delta/2 < 3\delta/4 < 3\varepsilon/8$ of S. This completes the proof.

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