IMPROVING THE INTERSECTIONS OF LINES AND SURFACES

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Dedicated to R. L. Wilder on his seventieth birthday.

Suppose S^2 is a topological 2-sphere and L^1 is a straight line in Euclidean 3-space E^3. If one had an easy proof of the following corollary, one could use the corollary to simplify the proof of the Approximation Theorem (Theorem 7 of [2]) and the Side Approximation Theorem (Theorem 16 of [6]).

COROLLARY TO THEOREM 2. *For each* $\varepsilon > 0$ *there exists a homeomorphism* $h: E^3 \to E^3$ *such that* $L^1 \cap h(S^2)$ *is finite and* $h = I$ *(identity) outside an* ε*-neighborhood of* $S^2 \cap L^1$.

We do not get a proof of the corollary that makes it useful in proving these theorems, since we use consequences of the Side Approximation Theorem to the hilt in proving Theorem 2, the basis for the corollary. There seems to be good reason for giving a proof of Theorem 2 even if it is complicated, since aside from its intrinsic interest, Theorem 2 is needed in proving some other interesting theorems, such as the result by McMillan (Theorem 2 of [8]) that each 2-sphere topologically embedded in an arbitrary 3-manifold has a neighborhood that can be embedded in E^3.

We denote the distance function by ρ. If f_1, f_2 are two maps of the space X into the metric space Y, we use $\rho(f_1, f_2)$ to denote the least upper bound $\rho(f_1(x), f_2(x))$ $(x \in X)$.

THEOREM 1. *Suppose* S_1, S_2 *are two 2-spheres in* E^3, S_2 *is tame,* $\varepsilon > 0$, *and* X *is a tame Sierpinski curve in* S_1 *such that each component of* $S_1 - X$ *has diameter less than* ε. *Then there exists an isotopy* h_t $(0 \leq t \leq 1)$ *of* E^3 *onto itself such that* $h_0 = I$, *the set* $S_2 \cap h_1(X)$ *is the union of a finite number of mutually exclusive simple closed curves each in the inaccessible part of* $h_1(X)$, $h_1(X)$ *locally lies on different sides of* S_2 *near these simple closed curves,* $h_t = I$ *outside the* ε*-neighborhood of* $S_1 \cap S_2$, *and* $\rho(h_t, I) < \varepsilon$.

Proof. Let δ be a positive number such that each component of $S_1 - X$ has diameter less than $\varepsilon - 4\delta$.

Let S_1' be a tame 2-sphere obtained by replacing each component of $S_1 - X$ by the interior of a tame disk homeomorphically so close to the component that the tame disk has diameter less than $\varepsilon - 4\delta$ and each point of $S_1' \cap S_2$ lies within δ of $S_1 \cap S_2$. The Approximation Theorem (Theorem 7 of [2]) tells us that this replacement is possible, and Theorem 8.2 of [7] assures us that S_1' is tame.

Since S_2 is tame, we suppose it is polyhedral.

Since S_1' is tame, there exists a homeomorphism $g: K^2 \times [-1, 1] \to E^3$, where K^2 is a polyhedral 2-sphere and $g(K^2 \times 0) = S_1'$. Using Theorem 9 of [3] or Theorem 2 of [9], we learn that g can be chosen to be locally piecewise linear on $K^2 \times (0, 1]$. Let f_t $(0 \leq t \leq 1)$ be an isotopy of $K^2 \times [-1, 1]$ onto itself that is fixed outside a small neighborhood of $g^{-1}(S_1' \cap S_2)$ and pushes a small neighborhood of $g^{-1}(S_1' \cap S_2)$ in $K^2 \times 0$ onto a polyhedron on the positive side of $K^2 \times 0$. By considering $g f_t g^{-1}$

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and putting the adjusted S_1' into general position with respect to S_2 near S_2, one finds that there exists an isotopy g_t $(0 \leq t \leq 1)$ of E^3 onto itself such that $S_2 \cap g_1(S_1')$ is the union of a finite collection $\{J_i\}$ of mutually exclusive simple closed curves along which S_2 and $g_1(S_1')$ cross, and such that $\rho(g_t, I) < \delta$, $g_0 = I$, and $g_t = I$ outside the δ-neighborhood of $S_2 \cap S_1'$. Note that each point of each J_i lies within δ of $S_1' \cap S_2$ and within 2δ of $S_1 \cap S_2$.

We would be finished with the proof of Theorem 1 if the J_i were in the inaccessible part of $h_1(X)$ rather than merely in $g_1(S_1')$. Since this is not necessarily the case, we modify g_1. Note that $g_1(S_1')$ is tame and the components of $g_1(S_1' - X)$ have diameters less than $\varepsilon - 2\delta$. The following lemma implies that there exists an isotopy h_t': $E^3 \to E^3$ such that $h_0' = I$, $h_t'(g_1(S_1')) = g_1(S_1')$, the J_i lie in the inaccessible part of $h_1' g_1(X)$, $\rho(h_t', I) < \varepsilon - 2\delta$, and $h_t' = I$ outside the $(\varepsilon - 2\delta)$-neighborhood of the union of the J_i. An isotopy satisfying the requirements of Theorem 1 is given by $h_t = h_t' g_t$.

The following is an engulfing theorem for Sierpinski curves, and it shows how Sierpinski curves can be made to swallow certain 1-dimensional sets.

LEMMA. *Suppose* S *is a 2-sphere,* $\varepsilon > 0$, X *is a Sierpinski curve in* S *such that each component of* $S - X$ *has diameter less than* ε, *and* Y *is a closed 1-dimensional subset of* S. *Then there exists an isotopy* h_t: $S \to S$ *such that* $h_0 = I$, Y *lies in the inaccessible part of* $h_1(X)$, *each orbit* $h(x, t)$ $(0 \leq t \leq 1)$ *has diameter less than* ε, *and* $h_t = I$ *outside an* ε*-neighborhood of* Y. *In fact, if* S *is a tame sphere in* E^3, h_t *may be extended to an isotopy of* E^3 *onto itself with the same properties.*

Proof. Let X' be a Sierpinski curve in the inaccessible part of X such that each component of $S - X'$ has diameter less than ε. To see that such an X' exists, use [10] to obtain a map f: $S \to S$ such that each $f^{-1}(x)$ is either an inaccessible point of X or the closure of a component of $S - X$. If one lets X'' be a Sierpinski curve in S that has small holes and misses the countably many points of S with nondegenerate inverses under f^{-1}, one can use $f^{-1}(X'')$ for X'. We let $h_t = I$ on X'.

If D is a component of $S - X'$ that misses Y, we permit h_t to be the identity on D. In the next four paragraphs we describe h_t on each component D that intersects Y.

It follows from [10] that there exists a map g_D of $\overline{D}$ onto itself such that $g_D | \operatorname{Bd} \overline{D} = I$ and each $g_D^{-1}(x)$ is either an inaccessible point of X or the closure of a component of $D \cap (S - X)$. Let W_D be the countable set of points of D with nondegenerate inverses under g_D^{-1}, and let h_D be a homeomorphism of $\overline{D}$ onto itself such that $h_D | \operatorname{Bd} \overline{D} = I$ and $W_D \cap h_D(D \cap Y) = 0$.

Let X_D be a Sierpinski curve in $\overline{D}$ that contains $\operatorname{Bd} \overline{D} \cup h_D(D \cap Y)$ but no point of W_D. The curve X_D is obtained by removing from $\overline{D}$ some small holes about the points of W_D, the holes being so small that they do not reach $h_D(D \cap Y)$. Note that $h_D^{-1}(X_D)$ contains $D \cap Y$, $g_D^{-1} | X_D$ is a homeomorphism, and $g_D^{-1}(X_D) \subset X$.

Let $h_1 = h_D^{-1} g_D$ on $g_D^{-1}(X_D)$. This h_1 is extended to take $\overline{D}$ homeomorphically onto itself. Note that $h_1(X) \supset D \cap Y$, since

$$h_1(X) \supset h_1(g_D^{-1}(X_D)) = h_D^{-1}(X_D) \supset D \cap Y.$$

It follows from [1] that the homeomorphisms $h_0 | \overline{D}$ and $h_1 | \overline{D}$ can be extended to an isotopy $h_t | \overline{D}$ that is fixed on $\operatorname{Bd} \overline{D}$. The collection of such $h_t | \overline{D}$ defines an isotopy h_t: $S \to S$.

If S is tamely embedded in E^3, we let δ be a positive number such that each $h(x \times [0, 1])$ has diameter less than $\varepsilon - 2\delta$, and we let $f: (S \times [-1, 1]) \to E^3$ be a homeomorphism such that $f(x \times 0) = x$ and $\operatorname{diam} f(x \times [-1, 1]) < \delta$ for each x. To extend h_t to E^3, we let

$$h_t = I \text{ on } E^3 - f(S \times [-1, 1]) \quad \text{and} \quad h_t f(x \times s) = f(h_{(1-s^2)t}(x) \times s).$$

THEOREM 2. *Suppose* S *is a* 2-*sphere and* A *is a tame arc in* E^3. *Then for each* $\varepsilon > 0$ *there is an isotopy* $h_t: E^3 \to E^3$ $(0 \leq t \leq 1)$ *such that* $h_0 = I$, A *intersects* $h_1(S)$ *at only a finite number of points and pierces it at each of these points,* $\rho(h_t, I) < \varepsilon$, *and* $h_t = I$ *outside the* ε-*neighborhood of* $S \cap A$.

Proof. We suppose without loss of generality that A is straight, $A \cap S$ is 0-dimensional, and S misses the ends of A.

Let $\{C_i\}$ be a finite collection of mutually exclusive solid right circular cylinders, each of diameter less than $\varepsilon/2$, and each with a subarc of A as an axis, such that no base of any of these cylinders intersects S but their interiors cover $S \cap A$. We suppose that neither end of A lies in a C_i.

Let δ be a positive number less than the radius of any C_i, the distance between any pair of C_i, the distance from S to the base of any C_i, and the distance from S to $A - \bigcup C_i$.

Let X be a tame Sierpinski curve in S such that each component of S - X has diameter less than $\delta/4$. That such a curve exists follows from [4].

Theorem 1 implies that there exists an isotopy $h_t': E^3 \to E^3$ $(0 \leq t \leq 1)$ such that $h_0' = I$, $\rho(h_t', I) < \delta/4$, $h_t' = I$ outside the $\delta/4$-neighborhood of $\bigcup \operatorname{Bd} C_i$, and $h_1'(X) \cap \bigcup \operatorname{Bd} C_i$ is the union of a finite collection $\{J_{ij}\}$ of mutually exclusive simple closed curves such that each J_{ij} lies in the inaccessible part of $h_1'(X)$ and the parts of $h_1'(X)$ on different sides of J_{ij} in S near J_{ij} lie on different sides of $\operatorname{Bd} C_i$. Note that $h_t'(S)$ misses the bases of the C_i and also $A - \bigcup C_i$. Each component of $h_1'(S - X)$ has diameter less than $3\delta/4$; therefore the only components of $\operatorname{Bd} C_i \cap h_1'(S)$ that could separate the bases of C_i from each other on $\operatorname{Bd} C_i$ are the J_{ij}.

In this paragraph we point out why there is an arc $p_i q_i$ on $\operatorname{Bd} C_i$ joining the points p_i, q_i of $A \cap \operatorname{Bd} C_i$ such that $p_i q_i \cap h_1'(S)$ is a finite set of points and each of these points is a point of a J_{ij} at which $p_i q_i$ crosses J_{ij}. A similar argument is given in the proof of Theorem 4.1 of [5], but for completeness we include the argument. Since J_{ij} is in the inaccessible part of $h_1'(X)$, we see that if $R_1, R_2, \cdots$ is a sequence of different components of $\operatorname{Bd} C_i \cap h_1'(S)$ converging to a set intersecting a J_{ij}, then the diameters of these components converge to 0. It follows from [10] that there exists a map $f_i: \operatorname{Bd} C_i \to \operatorname{Bd} C_i$ such that each $f_i^{-1}(x)$ $(x \in \operatorname{Bd} C_i)$ is either a continuum which is the union of a component of $\operatorname{Bd} C_i \cap h_1(S)$ other than a J_{ij} and all points that this component separates from p_i, or a point of $\operatorname{Bd} C_i - h_1'(S)$, or a point of a J_{ij}. Since $f(\operatorname{Bd} C_i \cap h_1'(S))$ is the union of the sets $f(J_{ij})$ and a 0-dimensional set, there exists an arc A_i on $\operatorname{Bd} C_i$ from $f(p_i)$ to $f(q_i)$ that intersects $f_i h_1'(S)$ at only a finite set of points, at each of which A_i crosses an $f_i(J_{ij})$. Consequently $p_i q_i = f_i^{-1}(A_i)$. Note that $p_i q_i$ pierces $h_1'(S)$ at each point of $p_i q_i \cap h_1'(S)$, since $h_1'(X)$ lies on different sides of $\operatorname{Bd} C_i$ near J_{ij}. (See Theorem 3.4 of [5].)

Let A' be the arc obtained from A by replacing each $A \cap C_i$ by $p_i q_i$. Note that A' intersects $h_1'(S)$ at only a finite number of points and pierces it at each of these points. Let h''_t: $E^3 \to E^3$ $(0 \leq t \leq 1)$ be an isotopy such that $h''_0 = I$, $h''_t = I$ outside the $\delta/2$-neighborhood of $\bigcup C_i$, and $h''_1(A') = A$. Then $h''_t h'_t = h_t$ is the required isotopy.

THEOREM 3. *Suppose* M^3 *is a* 3-*manifold (not necessarily compact, but without boundary),* K *is a locally tame finite graph in* M^3, C^2 *is a* 2-*complex (not necessarily compact),* h *is a homeomorphism of* C^2 *onto a closed subset of* M^3, *and* $\varepsilon > 0$. *Then there exists an isotopy* h_t: $M^3 \to M^3$ $(0 \leq t \leq 1)$ *such that* $h_0 = I$, $\rho(h_t, I) < \varepsilon$, $h_t = I$ *outside the* ε-*neighborhood of* $K \cap h(C^2)$, *and* $K \cap h_1 h(C^2)$ *is a finite set of points such that for each such point there is a* 2-*simplex* σ *of* C^2 *such that* K *pierces* $h_1 h(\sigma)$ *at the point.*

Proof. We suppose $h(C^2)$ is adjusted so that $K \cap h(C^2)$ is 0-dimensional and misses the image under h of the 1-skeleton of C^2, while K has a triangulation such that the set of the vertices of this triangulation misses $h(C^2)$. Since h_t will only move points of M^3 that are near the 1-simplexes of K under this triangulation but not near the 0-skeleton, we suppose with no loss of generality that K is a tame arc. Let U be an open subset of M^3 such that $K \subset U$ and U is homeomorphic to E^3.

It follows from Theorem 2.1 of [5] that for each point p of $K \cap h(C^2)$ there is a disk D_p and a 2-sphere S_p such that $p \in \operatorname{Int} D_p \subset S_p \subset U$ and D_p lies in the image under h of a 2-simplex of C^2. Let $D_1, D_2, \cdots, D_n$ be a finite collection of such disks such that each point $K \cap h(C^2)$ lies in the interior of one of the disks and no two of the disks intersect each other. Let S_i be the 2-sphere in U containing D_i.

Let $A_1, A_2, \cdots, A_m$ be a finite collection of mutually exclusive subarcs of K such that each point of $K \cap h(C^2)$ is an interior point of one of these arcs, no one of the arcs intersects two of the D_i, and the A_j intersecting D_i does not intersect $S_i - D_i$. If we apply Theorem 2 to get a suitable isotopy h_t^i that is the identity except near A_i, we obtain the required isotopy $h_t = h_t^1 h_t^2 \cdots h_t^m$.

In another paper, we use the techniques of the present paper to get the following general position theorem for arbitrary 2-manifolds embedded in a 3-manifold.

THEOREM 4. *Suppose* M^3 *is a* 3-*manifold and* M_1^2, M_2^2 *are compact* 2-*manifolds embedded in* M^3. *Then, for each positive* ε, *there exists an isotopy* h_t: $M^3 \to M^3$ $(0 \leq t \leq 1)$ *such that* $h_0 = I$, $\rho(h_t, I) < \varepsilon$, $h_t = I$ *outside the* ε-*neighborhood of* $M_1^2 \cap M_2^2$, *and the collection of nondegenerate elements of* $M_1^2 \cap h_1(M_2^2)$ *is a null-sequence of tame, simple closed curves whose union is dense in* $M_1^2 \cap h_1(M_1^2)$.

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