EXTENSIONS OF ALGEBRA HOMOMORPHISMS

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The Hahn-Banach Theorem [1, pp. 28-29] has a variant, called the Monotone Extension Theorem [2, p. 20], [3, Corollary 2.3], in which the subadditive, nonnegatively homogeneous function of the former theorem is replaced by a partial-ordering condition. The Hahn-Banach Theorem has been generalized by Vincent-Smith [5] to the extent that the scalar ring of real numbers may be replaced by any ring of real-valued continuous functions on an extremely disconnected compact Hausdorff space. In a similar fashion the Monotone Extension Theorem can be so generalized. We cite the Hahn-Banach and Monotone Extension Theorems in this greater generality below. In this paper, we shall prove analogues of these two theorems for algebras and rings instead of modules. Our Theorem 1 is the analogue of the Monotone Extension Theorem for commutative algebras. We sharpen this result in Theorem 2 by replacing the requirement of commutativity by a weaker condition, and then we give a comparable result for rings instead of algebras (Theorem 3). Then, in Theorem 4, we convert the partial-ordering condition of Theorem 2 into a subadditive, nonnegatively homogeneous function, which yields a curious analogue of the Hahn-Banach Theorem for algebras instead of modules. Finally we give several applications of Theorems 2, 3, and 4.

Throughout the paper, R denotes the field of real numbers, X the partially ordered ring of real-valued continuous functions on some extremely disconnected compact Hausdorff space, and Y a subring of X. We consider X as an algebra over Y. We use the customary definitions of partially ordered modules, rings, and algebras, except that we do *not* require antisymmetry. An element a of a partially ordered module, ring, or algebra A is *positive* if $0 \leq a$, and the set of all positive elements of A (sometimes called the *positive wedge* of A) is denoted by A^+. A subset B of a partially ordered set A is *cofinal in* A if, for every $a \in A$, there exists some $b \in B$ such that $a \leq b$. If $\leq$ and $\leq'$ denote two partial orderings of a module, ring, or algebra A, then $\leq$ is called *finer than* $\leq'$ if $0 \leq' a$ implies $0 \leq a$, for all $a \in A$.

HAHN-BANACH THEOREM. *Let* B *be a submodule of a module* (A, +, .) *over* X, *and let* $P: A \to X$ *satisfy the conditions*

$$P(\alpha a) = \alpha P(a) \quad \textit{and} \quad P(a+b) \leq P(a) + P(b)$$

for all $\alpha \in X^+$ *and* $a, b \in A$. *If* T: (B, +, .) $\to$ (X, +, .) *is a homomorphism such that* $T \leq P$, *then there exists a homomorphism* T^*: (A, +, .) $\to$ (X, +, .) *that extends* T *and satisfies the inequality* $T^* \leq P$.

MONOTONE EXTENSION THEOREM. *Let* B *be a submodule of a partially ordered module* (A, +, . , $\leq$) *over* Y *such that* B^+ *is cofinal in* A (*or if* X = R, B^+ *is cofinal in* A^+). *If* T: (B, +, . , $\leq$) $\to$ (X, +, . , $\leq$) *is a homomorphism, then there exists a homomorphism* T^*: (A, +, . , $\leq$) $\to$ (X, +, . , $\leq$) *that extends* T.

Since the proofs of these theorems do not require significantly more technique than is involved in the proof of the classical Hahn-Banach Theorem, we leave the proofs for the reader.

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THEOREM 1. *Let* B *be a subalgebra of a partially ordered commutative algebra* $(A, +, \cdot, ., \leq)$ *over* Y *such that* B^+ *is cofinal in* A. *If*

$$T: (B, +, \cdot, ., \leq) \to (X, +, \cdot, ., \leq)$$

is a homomorphism, then there exists a homomorphism

$$T^*: (A, +, \cdot, ., \leq) \to (X, +, \cdot, ., \leq)$$

that extends T.

Proof. First of all, we may assume that A has a positive identity, say e, which is also the identity of the subalgebra B, and that T(e) is the identity of X. For we can adjoin an identity e to A in the usual manner, and then extend the ordering of A by the rule

$$0 \leq a + \alpha e + ne \quad \text{if } 0 \leq a,\ 0 \leq \alpha, \text{ and } 0 \leq n,$$

for all $a \in A$, $\alpha \in Y$ and all integers n. Then we can extend T to the algebra generated by B and e by the equation $T(b + \alpha e + ne) = T(b) + \alpha + n$, for all $b \in B$, $\alpha \in Y$ and all integers n.

Let

$$\mathscr{C} = \{(C, T_C): C \text{ is a subalgebra of } A \text{ and } T_C: (C, +, \cdot, ., \leq) \to (X, +, \cdot, ., \leq) \text{ is a homomorphism}\}.$$

Partially order $\mathscr{C}$ by the condition that $(C, T_C) \leq (D, T_D)$ whenever $C \subseteq D$ and T_D is an extension of T_C. By Zorn's Lemma, there is a maximal element of $\mathscr{C}$, say (C, T_C), such that $(B, T) \leq (C, T_C)$. Abbreviate T_C by T. Suppose that $C \neq A$. Let $z \in A - C$, and let D be the subalgebra of A generated by C and z. Define

$$\Gamma = \{T(a)/T(b): a \in C,\ b \in C^+,\ T(b) \text{ invertible},\ a \leq bz\}.$$

By the cofinality condition, there exist $c, d \in C$ such that $c \leq z \leq d$. Hence $T(c) \in \Gamma$ and T(d) is an upper bound of Γ. Since X is boundedly complete, sup Γ exists. Therefore, let $\gamma = \sup \Gamma$.

Let P(n) be the proposition: If $0 \leq \sum_0^n a_i z^i$, where $a_i \in C$, then $0 \leq \sum_0^n T(a_i)\gamma^i$. Assume that P(n) is true for all n. Then the function $T^*: (D, +, \cdot, ., \leq) \to (X, +, \cdot, ., \leq)$ given by the equation

$$T^*\left(\sum_0^n a_i z^i\right) = \sum_0^n T(a_i)\gamma^i$$

is a well-defined homomorphism. This contradicts the maximality of (C, T_C). Therefore, $C = A$, and T_C is the desired extension of T.

It remains to show that P(n) is true for all positive integers n. Let $0 \leq a_0 + a_1 z$, where $a_0, a_1 \in C$. Let $c \in C^+$ be such that $a_1 \leq c$, and let $b = c + e$. Thus $a_1 \leq b$ and T(b) is invertible. Let $\delta \in \Gamma$, so that $\delta = T(p)/T(q)$ for some $p \in C$ and $q \in C^+$ such that $p \leq qz$. Then

$$(b - a_1)p \leq (b - a_1)qz \leq a_0 q + bqz\,.$$

By the definition of γ, we have the relation $T((b - a_1)p - a_0 q)/T(bq) \leq \gamma$; replacing $T(p)/T(q)$ by δ, we find that $T(b - a_1)\delta \leq T(a_0) + T(b)\gamma$. Therefore

$$0 = \sup_{\delta \in \Gamma} T(b - a_1)(\delta - \gamma) \leq T(a_0) + T(a_1)\gamma\,.$$

This establishes the truth of P(1).

Assume that P(n) is true for some $n \geq 1$. Let $0 \leq \sum_0^{n+1} a_i z^i$, where $a_i \in C$. By the cofinality condition, there exists $b \in C^+$ such that $a_{n+1} \leq b$, and there exist $c, d \in C$ such that $c \leq z \leq d$. Let $\delta \in \Gamma$, so that $\delta = T(p)/T(q)$ for some $p \in C$ and $q \in C^+$ such that $p \leq qz$. Then

$$0 \leq \sum_0^{n+1} a_i qz^i + (qz - p)(b - a_{n+1})(z - c)^n + (qz - p)b(d - z)(z - c)^{n-1}\,.$$

After cancellation, this inequality contains no term in z^{n+1}. It follows from P(n) that

$$0 \leq \sum_0^{n+1} T(a_i q)\gamma^i + (T(q)\gamma - T(p))\beta\,,$$

where

$$\beta = T(b - a_{n+1})(\gamma - T(c))^n + T(b)(T(d) - \gamma)(\gamma - T(c))^{n-1}\,.$$

After dividing by $T(q)$ and replacing $T(p)/T(q)$ by δ, we obtain the inequality $0 \leq \sum_0^{n+1} T(a_i)\gamma^i + (\gamma - \delta)\beta$. Thus

$$0 = \sup_{\delta \in \Gamma} (\delta - \gamma)\beta \leq \sum_0^{n+1} T(a_i)\gamma^i\,.$$

Therefore $P(n + 1)$ is true. This completes our proof.

Remark 1. If A, B, and T satisfy the hypotheses of Theorem 1, and if $z \in A - B$, then, by the techniques used in the proof of Theorem 1, we can construct our extension $T^*\colon A \to X$ in such a way that

$$T^*(z) = \sup\{T(a)/T(b)\colon a \in B,\ b \in B^+,\ T(b) \text{ invertible},\ a \leq bz\}\,.$$

Alternatively, we may construct T^* so that

$$T^*(z) = \inf\{T(a)/T(b)\colon a \in B,\ b \in B^+,\ T(b) \text{ invertible},\ a \geq bz\}\,.$$

In general, these are the only two admissible values that can be assigned to z to enlarge the homomorphism T. For example, let $A = R \times R$ (with the cartesian-product definitions of $+, \cdot, .\,, \leq$), let B be the diagonal, and let $T\colon B \to R$ be given by $T((\alpha, \alpha)) = \alpha$. Then the numbers 0 and 1 are the only admissible values that can be assigned to the point (0, 1) to enlarge the homomorphism T.

Remark 2. The cofinality condition in Theorem 1 is not superfluous. In fact, it cannot even be weakened to the condition that B^+ be cofinal in A^+. For example, let A be the partially ordered algebra of complex numbers over R with positive wedge $A^+ = R^+$, let $B = X = R$, and let T be the identity on B. Although B^+ is cofinal in A^+, T cannot be extended to A.

Remark 3. The commutativity condition in Theorem 1 is not superfluous. For example, let A be the partially ordered algebra of 2-by-2 matrices over R, where A^+ is the set of all matrices whose entries are all nonnegative. Let B be the subalgebra generated by [1] (the matrix whose entries all equal 1), and let $T: B \to R$ be defined by $T(\alpha[1]) = 2\alpha$. The algebras A and B and the homomorphism T satisfy all hypotheses of Theorem 1 except that A is not commutative. However, T cannot be extended to A. For whereas $T([1]) = 2$, the matrix [1] is the sum of one idempotent and two nilpotent elements.

Remark 4. The commutativity condition in Theorem 1 can be relaxed to the condition that $ab \leq ba$ for all $a, b \in A$; for we can apply Theorem 1 to the partially ordered algebra A' consisting of all equivalence classes of A, where a and b belong to the same equivalence class whenever $a \leq b \leq a$. This particular variant of Theorem 1 will be useful in the proof of the next theorem, which allows the commutativity of A to be dispensed with whenever B is in the center of A.

THEOREM 2. *Let* B *be a subalgebra in the center of a partially ordered algebra* A *over* Y *such that* B^+ *is cofinal in* A. *If* $T: (B, +, \cdot, ., \leq) \to (X, +, \cdot, ., \leq)$ *is a homomorphism, then there exists a homomorphism*

$$T^*: (A, +, \cdot, ., \leq) \to (X, +, \cdot, ., \leq)$$

that extends T.

Proof. As in the proof of Theorem 1, we may assume that A has an identity, say e, which is in B, and that T(e) is the identity of X. The following two useful properties follow from the cofinality condition:

[*] If $a \in A$, there exist elements $b \in B^+$ and $c \in A^+$ such that $a = b - c$.

[**] If $\leq'$ is finer than $\leq$, then $a \leq' b \leq' a$ implies $ac \leq' bc \leq' ac$ and $ca \leq' cb \leq' ca$, for all $a, b, c \in A$.

Let

$$\mathscr{C} = \{(C, \leq_C, T_C): C \text{ is a subalgebra of } A, \leq_C \text{ is a partial ordering on } A$$
$$\text{(compatible with } +, \cdot, . \text{) such that } ac \leq_C ca \leq_C ac \text{ for all } a \in A \text{ and}$$
$$c \in C, \text{ and } T_C: (C, +, \cdot, ., \leq_C) \to (X, +, \cdot, ., \leq) \text{ is a homomorphism}\}.$$

Partially order $\mathscr{C}$ by the condition that $(C, \leq_C, T_C) \leq (D, \leq_D, T_D)$ whenever $C \subseteq D$, $\leq_D$ is finer than $\leq_C$, and T_D is an extension of T_C. By Zorn's Lemma, there is a maximal element of $\mathscr{C}$, say $(C, \leq_C, T_C)$, such that

$$(B, \leq, T) \leq (C, \leq_C, T_C).$$

Suppose that $C \neq A$. Let $z \in A - C$, and let D be the subalgebra of A generated by C and z. Since each element of D has the form

$$\sum_{i=1}^{m} \prod_{j=1}^{n_i} x_{ij} \qquad (x_{ij} \in C \cup \{z\}),$$

one can easily show with the help of [**] that $ab \leq_C ba \leq_C ab$ for all $a, b \in D$. Thus, by Remark 4, there exists a homomorphism $T_D: (D, +, \cdot, ., \leq_C) \to (X, +, \cdot, ., \leq)$ that extends T_C, and furthermore, by Remark 1, we may require that $T_D(z) = \sup \Gamma$, where

$$\Gamma = \{T_C(a)/T_C(b)\colon a, b \in C,\ 0 \leq_C b,\ T_C(b) \text{ invertible},\ a \leq_C bz\}.$$

Let I be the ideal of A generated by all elements of the form $az - za$, where $a \in A$. Let $\leq_D$ be the partial ordering of A defined by the rule

$$0 \leq_D a \qquad \text{if } 0 \leq_C a + b \text{ for some } b \in I.$$

Clearly, the ordering $\leq_D$ is finer than the ordering $\leq_C$ and is compatible with the algebraic operations of A. From the facts that elements of D have the form displayed above and that $ab \leq_D ba \leq_D ab$ for all $a \in A$ and $b \in C \cup \{z\}$, we infer that $ab \leq_D ba \leq_D ab$ for all $a \in A$ and $b \in D$. Finally, we shall show that $0 \leq_D a$ implies $0 \leq T_D(a)$. This will complete our proof, because then we shall have the relation $(C, \leq_C, T_C) < (D, \leq_D, T_D) \in \mathscr{C}$, which contradicts the maximality of $(C, \leq_C, T_C)$.

Suppose that $0 \leq_D a$. By [*] and the definition of the ordering $\leq_D$, there exist $r_i, s_i, t_i, u_i, v_i, w_i \in A^+$ such that

$$0 \leq_C a + \sum_1^n r_i(s_i z - zs_i)t_i + \sum_1^n u_i(zv_i - v_i z)w_i.$$

There exist $b_i, c_i, d_i, f_i, g_i, h_i \in B^+$ such that

$r_i \leq b_i,\quad s_i \leq c_i,\quad t_i \leq d_i,\quad u_i \leq f_i,\quad v_i \leq g_i,\quad w_i \leq h_i$ for each i between 1 and n.

Let $\delta \in \Gamma$, so that $\delta = T_C(p)/T_C(q)$ for some $p, q \in C$ such that $0 \leq_C q$ and $p \leq_C qz$. Then

$$0 \leq_C qa + \sum_1^n r_i s_i(qz - p)t_i - \sum_1^n r_i(qz - p)s_i t_i + \sum_1^n u_i(qz - p)v_i w_i - \sum_1^n u_i v_i(qz - p)w_i$$

$$\leq_C qa + \sum_1^n r_i s_i(qz - p)t_i + \sum_1^n u_i(qz - p)v_i w_i \leq_C qa + (qz - p)\sum_1^n (b_i c_i d_i + f_i g_i h_i).$$

Hence

$$0 \leq T_D(qa) + (T_D(qz) - T_D(p))\,T_D\Big(\sum_1^n (b_i c_i d_i + f_i g_i h_i)\Big);$$

from which we derive the inequality

$$0 = \sup_{\delta \in \Gamma} (\delta - T_D(z)) T_D\left(\sum_1^n (b_i c_i d_i + f_i g_i h_i)\right) \leq T_D(a).$$

This completes the proof.

Since every partially ordered ring is a partially ordered algebra over the integers of X, Theorem 2 yields the following special case:

THEOREM 3. *Let* B *be a subring in the center of a partially ordered ring* $(A, +, \cdot, \leq)$ *such that* B^+ *is cofinal in* A. *If* $T: (B, +, \cdot, \leq) \to (X, +, \cdot, \leq)$ *is a homomorphism, then there exists a homomorphism* $T^*: (A, +, \cdot, \leq) \to (X, +, \cdot, \leq)$ *that extends* T.

In view of the close connection between the Monotone Extension Theorem and the Hahn-Banach Theorem, we might expect that Theorem 2 is closely connected with a theorem of the Hahn-Banach type for algebras. Indeed this is the case, as we now show.

THEOREM 4. *Let* B *be a subalgebra in the center of an algebra* $(A, +, \cdot, .)$ *over* X, *and let* $P: A \to X$ *be a function satisfying the conditions*

$$P(\alpha a) = \alpha P(a), \quad P(a+b) \leq P(a) + P(b), \quad P(P(a)b + P(b)a - ab) \leq P(a)P(b)$$

for all $\alpha \in X^+$ *and* $a, b \in A$. *If* $T: (B, +, \cdot, .) \to (X, +, \cdot, .)$ *is a homomorphism such that* $T \leq P$, *then there exists a homomorphism* $T^*: (A, +, \cdot, .) \to (X, +, \cdot, .)$ *that extends* T *and satisfies the inequality* $T^* \leq P$.

Proof. Let C be the algebra $(A \times X, +, \cdot, .)$, where addition and scalar multiplication have the usual product definitions and where

$$(a, \alpha)\cdot(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta) \qquad \text{for all } \alpha, \beta \in X \text{ and } a, b \in A.$$

We can make C into a partially ordered algebra by defining $\leq$ by the rule $0 \leq (a, \alpha)$ if $P(-a) \leq \alpha$. To see that $C^+ \cdot C^+ \subseteq C^+$, let $(a, \alpha), (b, \beta) \in C^+$, so that $P(-a) \leq \alpha$ and $P(-b) \leq \beta$. Then

$$\begin{aligned} P(-\alpha b - \beta a - ab) &\leq P((P(-a) - \alpha)b) + P((P(-b) - \beta)a) + P(-P(-a)b - P(-b)a - ab) \\ &\leq (\alpha - P(-a))P(-b) + (\beta - P(-b))P(-a) + P(-a)P(-b) \leq \alpha\beta. \end{aligned}$$

Hence $(a, \alpha)\cdot(b, \beta) \in C^+$. Let D be the subalgebra $B \times X$, and define $T: D \to X$ by the equation $T((a, \alpha)) = T(a) + \alpha$. Note that D is in the center of C, that T is an order-preserving algebra homomorphism, and that D^+ is cofinal in C because $(a, \alpha) \leq (0, |P(a) + \alpha|)$, for all $a \in A$ and $\alpha \in X$. Therefore, by Theorem 2, there exists a homomorphism $T^*: (C, +, \cdot, ., \leq) \to (X, +, \cdot, ., \leq)$ that extends T. Define $T^*: A \to X$ by the equation $T^*(a) = T^*((a, 0))$ for all $a \in A$. Since $(a, 0) \leq (0, P(a))$, we see that

$$T^*(a) = T^*((a, 0)) \leq T^*((0, P(a))) = P(a) \qquad \text{for all } a \in A;$$

consequently, $T^* \leq P$. Therefore, T^* is the desired extension of T.

Remark 5. The condition $P(P(a)b + P(b)a - ab) \leq P(a)P(b)$ in Theorem 4 seems to be the simplest additional condition sufficient to give a theorem of the Hahn-Banach type for algebras. The simpler condition $P(ab) = P(a)P(b)$ fails (hence the conditions $P(ab) \leq P(a)P(b)$ and $P(ab) \geq P(a)P(b)$) also fail), as one easily sees by letting A be the algebra of complex numbers over R, P the absolute-value function, and T the identity on R.

Example 1. Let $C^*(S)$ be the Banach algebra of all bounded continuous functions from S into R with the usual norm ($\|f\| = \sup\{|f(s)|: s \in S\}$). Let B be a subalgebra of $C^*(S)$. Then any continuous algebra homomorphism $T: B \to R$ can be extended to a continuous algebra homomorphism $T^*: C^*(S) \to R$ with equal norm. For assume that $T \neq 0$. Then, by using the Weierstrass Approximation Theorem, one can show. successively that $\|T\| = 1$ and that $T \leq P$, where P is given by the equation $P(f) = \sup\{f(s): s \in S\}$. (To obtain the inequality $T \leq P$, suppose that $P(f) < T(f)$ for some f. We may assume that $T(f) \neq 0$; for if $T(f) = 0$, consider instead the function $2T(g^2)f + P(f)g^2$, where $g \in B$ is such that $T(g) \neq 0$. Let $[\alpha, \beta]$ be an interval of R that contains 0, T(f), and the range of f, and let $F: [\alpha, \beta] \to R$ be continuous and satisfy the conditions

$$F(0) = 0, \qquad F(T(f)) = 1, \qquad F(\text{range } f) = 0.$$

There exists a polynomial G such that $G(0) = 0$ and $|G - F| \leq 1/3$ on $[\alpha, \beta]$. Then $T(G(f)) \geq 2/3$ and $\|G(f)\| \leq 1/3$, contradicting $\|T\| = 1$.) Since P and T satisfy the conditions of Theorem 4, there exists a homomorphism $T^*: C^*(S) \to R$ that extends T and satisfies the condition $T^* \leq P$. Thus $\|T^*\| = 1$.

The continuity of T is actually necessary as well as sufficient for the existence of an extension to $C^*(S)$, since each algebra homomorphism $T^*: C^*(S) \to R$ is continuous. Not every algebra homomorphism on a subalgebra of $C^*(S)$ need be continuous, however. For example, let B be the subalgebra of $C^*([0, 1])$ consisting of restrictions of polynomials to [0, 1], and let $T: B \to R$ be given by the equation $T(p) = p(2)$ for all $p \in B$.

Example 2. Let a and b denote arbitrary sequences of real numbers, and let e and e' be the sequences defined by $e(n) = 1$ for all n and

$$e'(0) = 1, \qquad e'(1) = -1, \qquad e'(n + 2) = 0 \text{ for all } n.$$

Let $(S, +, \cdot, .)$ be the algebra with identity e, over R, of all real-valued sequences, where multiplication is given by the equation $a \cdot b = a * b * e'$ and $*$ is the Cauchy product multiplication defined by the formula $a * b(n) = \sum_0^n a(i)b(n - i)$. Let A be the subalgebra of S generated by the space of all bounded sequences. We shall show that there exist Banach limits that can be expanded into algebra homomorphisms on the algebra A. To this end, we define a partial ordering $\leq_S$ on S by the rule that $0 \leq_S a$ if $0 \leq a(n)$ for all n. Because the ordering $\leq_S$ is not compatible with multiplication, we define a finer ordering that is compatible by the condition

$$0 \leq a \qquad \text{if } 0 \leq_S a * \left(\mathop{*}_{i=1}^{n} e\right) \text{ for some } n.$$

Let B be the subalgebra of A consisting of all sequences that are constant except for finitely many values. Let $T: B \to R$ be the order-preserving algebra homomorphism defined by $T(a) = \lim a(n)$. Because B^+ is cofinal in A, T has an extension to A, by Theorem 2. Call this extension T. If σ is the shift operator, that is, if $\sigma(a)(n) = a(n + 1)$ for all $a \in A$ and all n, then $T = T \circ \sigma$. This follows from the fact that

$$a = \sigma(a)\cdot\pi_0(e) + \tau_{a(0)}(0) \qquad \text{for all } a \in A,$$

where τ_α is the operator defined by the conditions $\tau_\alpha(a)(0) = \alpha$ and

$$\tau_\alpha(a)(n+1) = a(n) \qquad \text{for all } n.$$

Also, if $\alpha = \sup_n |a(n)|$ exists, then $a \leq \alpha e$, and hence $T(a) \leq \alpha$. Consequently, the restriction of T to the space of bounded sequences is a Banach limit.

Example 3. Let (F, +, *, .) be the algebra of formal power series over R in one indeterminate X. The algebra A in Example 2 is isomorphic to the subalgebra G of F generated by the space

$$\left\{ \sum_0^\infty \alpha_i X^i : \sup_n \left| \sum_0^n \alpha_i \right| < \infty \right\}$$

under the isomorphism that maps a into $(1 - X) * \left(\sum_0^\infty a(i)X^i\right)$. Therefore, as a consequence of Example 2, there is an algebra homomorphism $T: G \to R$ that satisfies the relation $T\left(\sum_0^\infty \alpha_i X^i\right) \leq \limsup_n \sum_0^n \alpha_i$, whenever the latter exists.

Example 4. We can achieve a result quite analogous to the preceding example, by using instead of (F, +, *, .) the algebra (L, +, *, .), where L is the set of all functions $f: R^+ \to R$ that are Lebesgue integrable on every finite interval, and where multiplication is defined by the formula $f * g(\alpha) = \int_0^\alpha f(\beta)g(\alpha - \beta)\,d\beta$.

Example 5. It is well known that every totally ordered division ring whose integers are cofinal is isomorphic to a subfield of R. In view of Theorem 3, this characterization can be sharpened to any partially ordered division ring whose integers are cofinal, provided the ordering is not the discrete or indiscrete ordering. To see this, let $(A, +, \cdot, \leq)$ be such a division ring, let Z be the set of integers of A, and let $T: Z \to R$ be the natural homomorphism defined by $T(n.1) = n$ (the symbol $n.1$ denotes the sum of 1 added to itself n times). By the ordering conditions, T is well-defined and preserves order. By Theorem 3, T has an extension T^* to A. Since A is a division ring, the kernel of T^* is $\{0\}$. Therefore, T^* is a ring isomorphism (but not necessarily an ordered-ring isomorphism) onto some subfield of R.

Example 6. Let $(A, +, \cdot, \leq)$ be a partially ordered ring with positive identity e. Then A is ordered-ring isomorphic to a subring of C*(S) for some (discrete) space S if and only if the following four conditions hold:

(i) the ordering $\leq$ is antisymmetric;

(ii) A is archimedean, in the sense that if $n.a \leq e$ for all positive integers n, then $a \leq 0$;

(iii) the integers of A are cofinal in A;

(iv) if $0 \leq n.a$, where n is positive, then $0 \leq a$.

Similarly, a partially ordered algebra $(A, +, \cdot, ., \leq)$ over R with positive identity e is ordered-algebra isomorphic to a subalgebra of C*(S) for some space S if and only if conditions (i), (ii), and (iii) hold.

We shall prove only the first assertion. Let conditions (i), (ii), (iii), and (iv) hold for a partially ordered ring A with positive identity e. Let $\{T_\xi: \xi \in S\}$ be the set of all nonzero ordered-ring homomorphisms from A into R. Define a homomorphism $F: A \to C^*(S)$ by the equation $F(a)(\xi) = T_\xi(a)$. We need to show that F is one-to-one and that F^{-1} preserves order. It suffices to show that if $0 \leq F(a)$, then $0 \leq a$. So suppose that $0 \leq F(a)$. By Theorem 3 and Remark 1, the natural mapping from the integers of A onto the integers of R can be extended to a homomorphism $T_\xi: A \to R$ such that

$$T_\xi(a) = \sup\{i/j: i, j \text{ integers}, 0 < j, i.e \leq j.a\}.$$

Since $0 \leq T_\xi(a)$, it follows that for each positive integer n there are integers i and j such that $0 < j$, $-1/n < i/j$, and $i.e \leq j.a$, and hence that $-j.e \leq jn.a$. By (iv), we get the inequality $-e \leq n.a$ for each positive integer n. Therefore, by (ii), $0 \leq a$.

Example 7. We shall use Example 6 to prove a representation theorem due to Stone [4] which states that if $(A, +, \cdot, ., \leq)$ is a partially ordered algebra over R with positive identity e such that conditions (i), (ii), and (iii) of Example 6 hold and such that A is complete in the norm defined by the formula

$$\|a\| = \inf\{\alpha \in R^+: -\alpha e \leq a \leq \alpha e\},$$

then A is ordered-algebra isomorphic to C(S) for some compact Hausdorff space S. By Example 6, A is ordered-algebra isomorphic to some subalgebra of C(S), where S indexes the set of all nonzero order-preserving algebra homomorphisms from A into R. By the Weierstrass Approximation Theorem and the completeness condition, A is isomorphic to a sublattice of C(S) under the same isomorphism. We topologize S by letting the sets

$$\{\xi \in S: T_\xi(a) \neq 0\} \qquad (a \in A)$$

be a basis for the topology. It is an easy exercise to show that the elements of A are mapped into continuous functions under the isomorphism defined in Example 6.

It remains to show that S is a compact Hausdorff space; for if this is the case, then A is isomorphic to all of C(S), by the Stone-Weierstrass Theorem. For every $\zeta, \xi \in S$ such that $\zeta \neq \xi$, there exists $a \in A$ such that $T_\zeta(a) = 0$ and $T_\xi(a) \neq 0$; for if $T_\zeta(a) = 0$ implies $T_\xi(a) = 0$ for all $a \in A$, then $\ker T_\xi$ is a maximal ideal containing the maximal ideal $\ker T_\zeta$, and hence $T_\zeta = T_\xi$. Let $\zeta, \xi \in S$ be such that $\zeta \neq \xi$. Then there exist $a, b \in A$ such that

$$T_\zeta(a) = 0, \quad T_\zeta(b) > 0, \quad T_\xi(a) > 0, \quad T_\xi(b) = 0.$$

The sets $\{\eta: T_\eta(b - a \wedge b) \neq 0\}$ and $\{\eta: T_\eta(a - a \wedge b) \neq 0\}$ are disjoint open sets containing ζ and ξ, respectively. Therefore, S is a Hausdorff space. Suppose the sets

$$\{\xi \in S: T_\xi(a_\lambda) = 0\} \qquad (a_\lambda \in A, \lambda \in \Lambda)$$

have the finite intersection property. Let T be the homomorphism from the algebra generated by the set $\{e, a_\lambda: \lambda \in \Lambda\}$ into R defined by $T(e) = 1$ and $T(a_\lambda) = 0$ for all $\lambda \in \Lambda$. By the finite intersection property, T is well-defined and preserves order. By Theorem 2, T has an extension to A, say T_ξ. Then ξ is in the total intersection of the sets displayed above. Therefore, S is compact.

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