MAXIMALLY ALMOST PERIODIC AND UNIVERSAL EQUICONTINUOUS MINIMAL SETS

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Let T be a topological group. A transformation group (X, T) with compact phase space X is called a *universal equicontinuous minimal set for* T if (X, T) is equicontinuous minimal and if each equicontinuous minimal transformation group (Y, T) with compact phase is a homomorphic image of (X, T). A metatheorem concerning universal "admissible" minimal sets [2, Theorem 2] guarantees the isomorphically unique existence of such a universal object for T.

Chu [3] has defined in a different way the notion of a universal almost periodic minimal set for T, called here a *maximally almost periodic minimal set for* T (to avoid ambiguity caused by the equivalence of equicontinuity and almost periodicity for a transformation group with compact phase space).

We discuss below the relations to each other and to group compactifications of T of these two types of objects. If a transformation group (X, T) is a universal equicontinuous minimal set for T, then it is a maximally almost periodic minimal set for T; we show that the converse holds if T is compact and (X, T) is effective. We prove the existence of many maximally almost periodic minimal sets for noncompact generative T that are not universal equicontinuous minimal sets for T. As general references on the category of minimal transformation groups, see [4] and [5].

All compact and locally compact spaces considered are assumed to be Hausdorff spaces.

For a topological space X, $C^*(X)$ denotes the Banach algebra of bounded continuous real-valued functions on X with the uniform norm, and ϕ^* denotes the canonical map of $C^*(Y)$ into $C^*(X)$ induced by a continuous map ϕ of X into a space Y.

Let T_d denote the group underlying T provided with its discrete topology. Let $(C^*(T), T_d, \lambda)$ be the transformation group of isometric automorphisms of $C^*(T)$ given by

$$s((f, t)\lambda) = (ts)f \qquad (t, s \in T;\ f \in C^*(T)).$$

Let $A^*(T)$ be the T_d-invariant subalgebra of $C^*(T)$ consisting of those functions in $C^*(T)$ that are (left) almost periodic, that is, whose orbits under λ are relatively compact. We denote by λ again the restriction of λ to $A^*(T) \times T_d$.

We call a couple (X, ϕ), where X is a compact space and ϕ is a continuous map of T into X, an *almost periodic compactification of* T if $T\phi$ is dense in X and $C^*(X)\phi^* = A^*(T)$, so that ϕ^* is an isometric isomorphism of $C^*(X)$ with $A^*(T)$. A *maximally almost periodic minimal set for* T is then an equicontinuous minimal transformation group (X, T) with compact phase space X such that (X, ϕ) is an almost periodic compactification of T for some ϕ. (Chu's definition of a universal almost periodic minimal set imposes an additional condition that is superfluous, in view of Corollary 2.) Clearly, any two maximally almost periodic minimal sets for

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T have homeomorphic phase spaces, but Corollary 1 and Theorem 3 show they need not be isomorphic as transformation groups.

A couple (G, ϕ) is called, as usual, a *group compactification of* T if G is a compact group and ϕ is a continuous homomorphism of T onto a dense subgroup of G, and (G, ϕ) is said to be *maximal* if for each group compactification (K, ψ) of T there exists a continuous homomorphism θ (necessarily unique) of G onto K with $\phi\theta = \psi$. Any two maximal group compactifications of T are isomorphic in the obvious sense, but the following construction seems most appropriate in the present context.

Let M(T) be the enveloping semigroup [4] of $(A^*(T), T_d, \lambda)$, and let μ_T, or simply μ, denote the map $t \to \lambda^t$ of T into M(T). It is well known that $(M(T), \mu)$ is both a maximal group compactification of T and an almost periodic compactification of T (see [7] for example).

Corresponding to a group compactification (G, ϕ) of T and a closed subgroup H of G, let $H \setminus G$ be the right coset space of H in G, and define $(H \setminus G, T, \bar{\phi})$ to be the transformation group such that

$$(Hg, t)\bar{\phi} = Hg(t\phi) \qquad (g \in G,\ t \in T).$$

When H is the trivial subgroup of G, $(H \setminus G, T, \bar{\phi})$ reduces to the right transformation group $(G, T, \bar{\phi})$ of G induced by T under ϕ [6, 1.58]. For example, $(M(T), T_d, \bar{\mu})$ is precisely the enveloping transformation group of $(A^*(T), T_d, \lambda)$.

LEMMA 1 (from [5, p. 57], [6, 4.48]). *A necessary and sufficient condition for a transformation group* (X, T, π) *with compact phase space to be equicontinuous minimal is that it be isomorphic to* $(H \setminus G, T, \bar{\phi})$ *for some group compactification* (G, ϕ) *of* T *and some closed subgroup* H *of* G. *In case* T *is abelian, an equivalent condition is that* (X, T, π) *be isomorphic to* $(G, T, \bar{\phi})$ *for some group compactification* (G, ϕ) *of* T.

Remark. Let (G, ϕ) be a group compactification of T, and for each $g \in G$ let τ_g be the left translation in G by g. Then $g \to \tau_g$ is an algebraic anti-isomorphism of G onto the group of automorphisms of $(G, T, \bar{\phi})$.

LEMMA 2. *Let* (G, ϕ) *and* (K, ψ) *be group compactifications of* T, *and let* θ *be a continuous homomorphism of* K *onto* G *with* $\psi\theta = \phi$. *Then*

(1) θ *is the unique homomorphism of* $(K, T, \bar{\psi})$ *onto* $(G, T, \bar{\phi})$ *mapping the identity of* K *onto the identity of* G,

(2) *a map* $\eta: K \to G$ *is a homomorphism of* $(K, T, \bar{\psi})$ *onto* $(G, T, \bar{\phi})$ *if and only if* $\eta = \theta\tau_g$ *for some* $g \in G$,

(3) *if* $(K, T, \bar{\psi})$ *is isomorphic to* $(G, T, \bar{\phi})$, *then* θ *is a homeomorphic isomorphism of* K *onto* G.

Proof. (1) Clearly, θ is equivariant with respect to the actions $\bar{\psi}$, $\bar{\phi}$. Let θ' be a homomorphism of $(K, T, \bar{\psi})$ onto $(G, T, \bar{\phi})$, with $e\psi\theta' = e\phi$, where e is the identity element of T. Then $(x \cdot t\phi)\theta' = (x\theta')(t\phi)$ for all $x \in K$, $t \in T$. Taking $x = e\psi$, we see that θ' coincides with θ on $T\psi$, whence $\theta' = \theta$.

(2) Let η be a homomorphism of $(K, T, \bar{\psi})$ onto $(G, T, \bar{\phi})$. If $g = (e\psi\eta)^{-1}$, then $\eta = \theta\tau_g$, by (1).

(3) follows from (2) and the preceding Remark.

THEOREM 1. *Let* (X, T, π) *be a transformation group with compact phase space. Then the following are equivalent:*

(1) (X, T, π) *is a universal equicontinuous minimal set for* T.

(2) (X, T, π) *is isomorphic to* $(M(T), T, \bar{\mu})$.

(3) *For each maximal group compactification* (G, ϕ) *of* T, (X, T, π) *is isomorphic to* $(G, T, \bar{\phi})$.

Proof. Let (K, ψ) be a maximal group compactification of T. Since any two universal equicontinuous minimal sets for T are isomorphic, it suffices to show that $(K, T, \bar{\psi})$ is a universal equicontinuous minimal set for T. Let (G, ϕ) be any group compactification of T, and let H be a closed subgroup of G. By Lemma 1, it is enough to show that $(H \setminus G, T, \bar{\phi})$ is a homomorphic image of $(K, T, \bar{\psi})$. Now the projection of G onto $H \setminus G$ is a homomorphism of $(G, T, \bar{\phi})$ onto $(H \setminus G, T, \bar{\phi})$, and $(G, T, \bar{\phi})$ is a homomorphic image of $(K, T, \bar{\psi})$, by Lemma 2 and the maximality of (K, ψ).

Since $(M(T), \mu)$ is an almost periodic compactification of T, Lemma 1 implies that $(M(T), T, \bar{\mu})$ is a maximally almost periodic minimal set for T. Hence we obtain the following three results.

COROLLARY 1. *Each universal equicontinuous minimal set for* T *is a maximally almost periodic minimal set for* T.

COROLLARY 2. *If* (X, T) *is a maximally almost periodic minimal set for* T, *and* (Y, T) *is an equicontinuous minimal transformation group with compact phase space, then* Y *is a continuous image of* X.

COROLLARY 3. *Let* (X, ϕ) *be an almost periodic compactification of* T. *Then there exists a unique action* π *of* T_d *on* X *such that*

$$(*) \qquad (st)\phi = (s\phi, t)\pi \qquad (s, t \in T);$$

moreover, (X, T, π) *is a universal equicontinuous minimal set for* T.

Proof. Let F be the inverse of the isomorphism of $C^*(X)$ onto $A^*(T)$ defined by ϕ^*. There is a unique homeomorphism θ of X onto $M(T)$, with $\theta^* = \mu^* F$, whence $\phi\theta = \mu$. Define $\pi: X \times T \to X$ by

$$(x, t)\pi = (x\theta, t)\bar{\mu}\, \theta^{-1} \qquad (x \in X,\ t \in T).$$

Then (X, T, π) is a transformation group isomorphic under θ to $(M(T), T, \bar{\mu})$, and direct computation gives (*). The stated uniqueness of π follows from the fact that if $t \in T$ and τ is a homeomorphism of X onto X with $(st)\phi = s\phi\tau$ for all $s \in T$, then τ and π^t agree on $T\phi$.

Remark. Let (G, ϕ) be a group compactification of T. Then (G, ϕ) is a maximal group compactification of T if and only if it is an almost periodic compactification of T (and in this case $\bar{\phi}$ is the action π given by Corollary 3). If (G, ψ) is an almost periodic compactification of T for *some* map ψ, there is no guarantee that (G, ϕ) is itself an almost periodic compactification of T.

If T is maximally almost periodic, that is, if $A^*(T)$ separates points of T, then μ is injective, and by the next lemma, $(M(T), T, \bar{\mu})$ is strongly effective.

LEMMA 3. *Let* (G, ϕ) *be a group compactification of* T. *Then* $(G, T, \bar{\phi})$ *is strongly effective if and only if* ϕ *is injective.*

THEOREM 2. *Suppose* T *is compact, and let* (X, T, π) *be a maximally almost periodic minimal set for* T. *Then a necessary and sufficient condition for* (X, T, π) *to be a universal equicontinuous minimal set for* T *is that it be effective.*

Proof. The necessity follows from Theorem 1 and Lemma 3. We show the sufficiency. We may suppose $(X, T, \pi) = (H \backslash G, T, \bar{\phi})$, where G, ϕ, H are as in Lemma 1. By the compactness of T, ϕ is surjective. Since $\bar{\phi}$ is a universally transitive action, it is strongly effective. Therefore $t\phi \notin H$ for all $t \in T$ with $t \neq e$, $H = \{e\}$, and ϕ is injective. Hence ϕ is an isomorphism of (T, T, τ) with $(G, T, \bar{\phi})$, where τ is the group operation in T. But μ is an isomorphism of (T, T, τ) with $(M(T), T, \bar{\mu})$.

Example. Let G be a nondegenerate compact group, let I be an infinite set, and let $T = G^I$. Choose an injection $\sigma: I \to I$ that is not surjective. Define ϕ to be the continuous "shift" endomorphism of T given by

$$(g_i \mid i \in I)\phi = (g_{i\sigma} \mid i \in I).$$

Then ϕ is surjective but not injective, and therefore $(T, T, \bar{\phi})$ is a maximally almost periodic minimal set for T that is not a universal equicontinuous minimal set for T.

For a locally compact abelian group G, G' denotes the character group of G, and $\phi': H' \to G'$ denotes the adjoint of a continuous homomorphism ϕ of G into a locally compact abelian group H; we sometimes identify G with its second character group G'' under duality.

Let T be locally compact and abelian. If G is the universal Bohr compactification $((T')_d)'$ of T, and $\phi: T \to G$ is the adjoint of the identity map of $(T')_d$ into T', then (G, ϕ) is both a maximal group compactification and an almost periodic compactification of T [1]. Hence we identify (G, ϕ) with $(M(T), \mu)$.

THEOREM 3. *Let* T *be generative (locally compact, compactly generated, abelian) and noncompact. Then there exist uncountably many pairwise nonisomorphic, strongly effective, maximally almost periodic minimal sets for* T *that are not universal equicontinuous minimal sets for* T.

Proof. By a structure theorem of Weil [8, p. 110], T has the form $R^m \times Z^n \times K$, where R is the line group, Z is the discrete group of integers, m and n are nonnegative integers, and K is compact and abelian. Since T is not compact, $T = V \times S$, where $V = R$ or $V = Z$ and S is locally compact and abelian. Identify $M(T)$ with $M(V) \times M(S)$, so that $\mu_T = \mu_V \times \mu_S$.

Let H be a Hamel basis for R_d, with $1 \in H$. There exists an uncountable collection $\mathscr{B}$ of subsets of H such that

(1) $\mathscr{B}$ is totally ordered under inclusion,

(2) each member of $\mathscr{B}$ is equipotent to H,

(3) $H \in \mathscr{B}$,

(4) $B \in \mathscr{B}$ implies $1 \in B$.

For each $B \in \mathscr{B}$, let $D(B)$ be the Q-submodule of R_d generated by B, where Q is the field of rationals.

Consider first the case $V = R$. Identify R' with R, as usual, so that μ_V is the adjoint of the identity map $i: R_d \to R$. For each $B \in \mathscr{B}$, let

$$G(B) = D(B)' \times M(S), \qquad \phi_B = (j_B\, i)' \times \mu_S: T \to G(B),$$

where $j_B: D(B) \to R_d$ is the inclusion map. Then $(G(B), \phi_B)$ is a group compactification of T for each $B \in \mathscr{B}$. In particular, $(G(H), \phi_H) = (M(T), \mu_T)$.

Let $B \in \mathscr{B}$. By Kronecker's approximation theorem, the image of D(B) under $j_B i$ is dense in R, whence $(j_B i)'$ is a monomorphism. Since μ_S is injective, it follows from Lemma 3 that $(G(B), T, \bar{\phi}_B)$ is strongly effective. By (2), D(B) is isomorphic with R_d. Therefore D(B)' is homeomorphically isomorphic with M(R), and G(B) is homeomorphic with M(T). Hence $(G(B), T, \bar{\phi}_B)$ is a maximally almost periodic minimal set for T.

Let $A, B \in \mathscr{B}$, with $A \neq B$. We show that $(G(A), T, \bar{\phi}_A)$ is not isomorphic with $(G(B), T, \bar{\phi}_B)$. In view of (1), we may assume $B \subset A$. If $k: D(B) \to D(A)$ is the inclusion map, and i_S is the identity map of M(S), then $k' \times i_S$ is a continuous homomorphism of G(A) onto G(B), with $\phi_A(k' \times i_S) = \phi_B$. Since k is not surjective, $k' \times i_S$ is not injective. Now apply Lemma 2.

Taking A = H above, we conclude that $(G(B), T, \bar{\phi}_B)$ is not isomorphic with $(M(T), T, \bar{\mu})$ for any $B \in \mathscr{B}$ distinct from H.

Suppose now that V = Z. Then μ_V is the adjoint of the identity map $j: R_d/Z \to R/Z$. For each $B \in \mathscr{B}$, $D(B) \supset Z$ by (4), and we take

$$G(B) = (D(B)/Z)' \times M(S), \qquad \phi_B = (k_B j)' \times \mu_S : T \to G(B),$$

where k_B is the inclusion map of D(B)/Z into R_d/Z. The arguments for the case V = R may now be repeated.

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