ACTIONS OF θ_{2k+1}

Rodolfo De Sapio

In this note we give a simple geometric proof that the group θ_{2k+1} of homotopy $(2k+1)$-spheres acts nontrivially on $S^k \times S^{k+1}$, the product of the k-sphere S^k with the $(k+1)$-sphere S^{k+1} (Lemma 1 below). More generally, let M^{2k+1} be a closed, oriented, infinitely differentiable $(2k+1)$-manifold that is $(k-1)$-connected and almost parallelizable. Suppose also that $k \equiv 2, 4, 6 \pmod 8$, and that Σ^{2k+1} is a homotopy $(2k+1)$-sphere such that the connected sum $M \# \Sigma$ is diffeomorphic to M; then Σ bounds a π-manifold (Lemma 2 below) and hence $\Sigma \# \Sigma$ is diffeomorphic to S^{2k+1}.

These results are of interest in the following connection. Let M^{2k+1} satisfy the conditions above, and let K be even. Then, by the results of Tamura [6] and Vasquez [7], M^{2k+1} is diffeomorphic to a connected sum

(1) $$(S^k \times S^{k+1}) \# \cdots \# (S^k \times S^{k+1}) \# V_{k+2,2} \# \cdots \# V_{k+2,2} \# M_T,$$

where $V_{k+2,2}$ is the tangent k-sphere bundle to the $(k+1)$-sphere and M_T is a $(k-1)$-connected manifold such that $H_k(M_T)$ is a torsion group that is not finite cyclic (this had also been shown by the author independently; however, according to [7], M_T also decomposes into a connected sum of rather simple manifolds). In particular, if $H_k(M)$ is free, then M_T in (1) is replaced by a homotopy $(2k+1)$-sphere. Thus we apply the result on the action of θ_{2k+1} on $S^k \times S^{k+1}$ to obtain the following result. (We use $[\theta_{2k+1}]$ to denote the order of the finite group θ_{2k+1}; see [4] for computations of these orders.)

PROPOSITION 1. *Let* $k \equiv 2, 4, 6 \pmod 8$, *with* $k > 2$ *and* $k \neq 6$. *Then the number of distinct (nondiffeomorphic) almost parallelizable,* $(k-1)$-*connected* $(2k+1)$-*manifolds* M^{2k+1} *with* $H_k(M)$ *cyclic and not zero is either*

(i) *exactly* $\frac{3}{2}[\theta_{2k+1}]$ (*the case where* $bP_{2k+2} \neq 0$),

or

(ii) *exactly* $2[\theta_{2k+1}]$ (*the case where* $bP_{2k+2} = 0$).

Furthermore, the manifolds are all of the form

$$(S^k \times S^{k+1}) \# \Sigma^{2k+1} \quad or \quad V_{k+2,2} \# \Sigma^{2k+1},$$

where Σ^{2k+1} *is a homotopy* $(2k+1)$-*sphere. In either case, there are exactly* $[\theta_{2k+1}]$ *distinct manifolds of the form* $(S^k \times S^{k+1}) \# \Sigma^{2k+1}$.

Here bP_{2k+2} is the subgroup of θ_{2k+1} of those homotopy spheres that bound π-manifolds. According to [4], bP_{2k+2} is either zero or of order two when k is even, and it follows from a result of [2] that, for $k \equiv 0 \pmod 4$, bP_{2k+2} is of order two. Hence for $k \equiv 0 \pmod 4$ we have conclusion (i) in the above proposition. It is a conjecture that bP_{2k+2} is of order two for all even integers $k > 2$, and hence that (i) is always true.

Received March 21, 1966.
This work was supported in part by National Science Foundation Grants No. 4069 and GP-5860.

The above proposition is false for the exceptional case $k = 6$, as has been remarked by the referee. In this case $V_{k+2,2}$ and $S^k \times S^{k+1}$ are diffeomorphic, since S^7 is parallelizable. Hence the number of distinct almost parallelizable, 5-connected 13-manifolds M^{13} with $H_6(M^{13})$ cyclic and not zero is exactly $[\theta_{13}] = 3$. The manifolds are all of the form $(S^6 \times S^7) \# \Sigma^{13}$, where Σ^{13} is a homotopy 13-sphere. However, in general the manifolds $V_{k+2,2}$ and $S^k \times S^{k+1}$ have different homotopy types. In fact, a result of I. M. James and J. H. C. Whitehead states that $V_{k+2,2}$ is of the same homotopy type as $S^k \times S^{k+1}$ if and only if $\pi_{2k+3}(S^{k+2})$ has an element with Hopf invariant one. But G. W. Whitehead and J. F. Adams have proved that $\pi_{2k+3}(S^{k+2})$ does not have an element with Hopf invariant one, provided that $k \neq 0, 2, 6$.

The proof of the proposition goes as follows. According to (1), M^{2k+1} is diffeomorphic either to $(S^k \times S^{k+1}) \# \Sigma^{2k+1}$ or to $V_{k+2,2} \# \Sigma^{2k+1}$, for some homotopy sphere Σ^{2k+1}. We shall now show that there are exactly $[\theta_{2k+1}]$ distinct manifolds of the form $(S^k \times S^{k+1}) \# \Sigma^{2k+1}$.

LEMMA 1. *If* Σ^{2k+1} *is a homotopy sphere such that* $(S^k \times S^{k+1}) \# \Sigma^{2k+1}$ *is diffeomorphic to* $S^k \times S^{k+1}$, *then* Σ^{2k+1} *is diffeomorphic to the standard* $(2k+1)$-*sphere* S^{2k+1}.

Proof. Let $h: (S^k \times S^{k+1}) \# \Sigma^{2k+1} \approx S^k \times S^{k+1}$ be a diffeomorphism, and let $p_0 \in S^{k+1}$. We may view the k-sphere $S^k \times p_0$ as being embedded in both $S^k \times S^{k+1}$ and $(S^k \times S^{k+1}) \# \Sigma^{2k+1}$ (this connected sum is made far away from the sphere $S^k \times p_0$). Furthermore, by standard arguments (theorems of Haefliger, and diffeotopy extension), and by composing the diffeomorphism h with the diffeomorphism of $S^k \times S^{k+1}$ that reverses the orientation of each factor of $S^k \times S^{k+1}$ if necessary, we may suppose that h is the identity on the k-sphere $S^k \times p_0$. Next, let $S^k \times D^{k+1}$ denote the "standard" product structure on $S^k \times p_0$ in both $(S^k \times S^{k+1}) \# \Sigma^{2k+1}$ and $S^k \times S^{k+1}$. By the tubular-neighborhood theorem of Milnor, we may further suppose that h maps $S^k \times D^{k+1} \subset (S^k \times S^{k+1}) \# \Sigma^{2k+1}$ onto $S^k \times D^{k+1} \subset S^k \times S^{k+1}$ in such a way that for each $(u, v) \in S^k \times D^{k+1}$, $h(u, v) = (u, v \cdot \alpha(u))$, where $\alpha: S^k \to SO_{k+1}$ is a smooth map and $v \cdot \alpha(u)$ denotes the action of $\alpha(u) \in SO_{k+1}$ on $v \in D^{k+1}$. Now perform the spherical modification on $(S^k \times S^{k+1}) \# \Sigma^{2k+1}$ that removes the k-sphere $S^k \times p_0$ with product structure $S^k \times D^{k+1}$ in $(S^k \times S^{k+1}) \# \Sigma^{2k+1}$ (note that we are really modifying the left-hand summand $S^k \times S^{k+1}$ of $(S^k \times S^{k+1}) \# \Sigma^{2k+1}$). The result of this modification is Σ^{2k+1}. We now perform the corresponding modification (under h) on $S^k \times S^{k+1}$ to remove the k-sphere $S^k \times p_0$ with product structure $h(S^k \times D^{k+1})$ in $S^k \times S^{k+1}$. From the latter modification we obtain the manifold

$$[(S^k \times S^{k+1}) - \text{Interior } h(S^k \times D^{k+1})] \cup_h [D^{k+1} \times S^k], \tag{2}$$

which is clearly diffeomorphic to Σ^{2k+1} because of the way we defined this modification (using h). It remains to show that the manifold (2) is diffeomorphic to S^{2k+1}. Write $S^k \times S^{k+1}$ as the union of two copies of $S^k \times D^{k+1}$, in the form

$$S^k \times S^{k+1} = (S^k \times D^{k+1})_1 \cup_{id} (S^k \times D^{k+1})_2$$

with points identified along the boundary $S^k \times S^k$ *via* the identity map id. Here $(S^k \times D^{k+1})_1$ is understood to be the standard product structure $S^k \times D^{k+1}$ on $S^k \times p_0$ in $S^k \times S^{k+1}$ introduced above. Then (2) may be written as

$$[(S^k \times D^{k+1})_2] \cup_h [D^{k+1} \times S^k], \tag{3}$$

and this is clearly diffeomorphic to

$$[(S^k \times D^{k+1})_2] \cup_{id} [D^{k+1} \times S^k], \tag{4}$$

by virtue of the map that sends $(u, v) \in (S^k \times D^{k+1})_2$ into

$$h(u, v) = (u, v \cdot \alpha(u)) \in (S^k \times D^{k+1})_2$$

and $(u, v) \in D^{k+1} \times S^k$ into (u, v) (this diffeomorphism goes from line (4) to line (3)). But (4) is diffeomorphic to S^{2k+1}, and this completes the proof of Lemma 1.

In order to complete the proof of the proposition, we must show that the number of distinct manifolds of the form $V_{k+2,2} \# \Sigma^{2k+1}$ (Σ^{2k+1} a homotopy $(2k+1)$-sphere) is $[\theta_{2k+1}]/2$ if bP_{2k+2} is of order two, and that it is $[\theta_{2k+2}]$ if bP_{2k+2} is zero, provided that $k \equiv 2, 4, 6 \pmod 8$. Now Brown and Steer [3] first proved that if Σ^{2k+1} bounds a π-manifold, then $V_{k+2,2} \# \Sigma^{2k+1}$ is diffeomorphic to $V_{k+2,2}$, provided k is even. This has also been proved by Kosinski [5], who has shown that the subgroup of θ_{2k+1} of the homotopy spheres Σ^{2k+1} for which $V_{k+2,2} \# \Sigma^{2k+1}$ is diffeomorphic to $V_{k+2,2}$ is a homomorphic image of $\pi_k(SO_{k+1})$ and contains bP_{2k+2}, provided that k is even (then for $k \equiv 4 \pmod 8$ it follows that this subgroup equals $bP_{2k+2} \approx Z_2$). Thus the following lemma implies that bP_{2k+2} is exactly the set of Σ^{2k+1} in θ_{2k+1} such that $V_{k+2,2} \# \Sigma^{2k+1}$ is diffeomorphic to $V_{k+2,2}$, from which the proposition follows.

LEMMA 2. *Let* $k \equiv 2, 4, 6 \pmod 8$. *If* M^{2k+1} *is an almost parallelizable,* $(k-1)$*-connected* $(2k+1)$*-manifold and* Σ^{2k+1} *is a homotopy* $(2k+1)$*-sphere such that* $M \# \Sigma$ *is diffeomorphic to* M, *then* Σ^{2k+1} *bounds a* π*-manifold.*

Remark. The following argument uses the fact that almost parallelizable manifolds M^{2k+1} of this special type are π-manifolds. Furthermore, this is true without any restriction on the integer k. In fact, the obstruction to the triviality of the stable tangent bundle of M^{2k+1} is a well-defined cohomology class

$$\sigma_{2k+1}(M) \in H^{2k+1}(M; \pi_{2k}(SO_{2k+2})) \approx \pi_{2k}(SO).$$

Now this group is zero for $k \not\equiv 0 \pmod 8$. For $k \equiv 0 \pmod 8$ it is known that $\sigma_{2k+1}(M)$ lies in the kernel of the Hopf-Whitehead homomorphism

$$J_{2k}: \pi_{2k}(SO_m) \to \pi_{2k+m}(S^m)$$

in the stable range $m > 2k+1$, and Adams has shown that J_{2k} is a monomorphism in this dimension. Hence the obstruction $\sigma_{2k+1}(M)$ is always zero.

Proof. For $k \equiv 4 \pmod 8$, the lemma follows from Kosinski [5]. By [4], M^{2k+1} may be reduced by framed spherical modifications to a homotopy $(2k+1)$-sphere Λ^{2k+1}, and hence, by replacing M^{2k+1} by $M^{2k+1} \# (-\Lambda^{2k+1})$ if necessary ($-\Lambda^{2k+1}$ is the manifold Λ^{2k+1} with the orientation reversed), we may assume that M^{2k+1} may be reduced to the standard sphere S^{2k+1} by framed modifications (thus M^{2k+1} bounds a π-manifold). Further, it is known that we may take each modification to be of type $(k+1, k+1)$. Hence we can assume that there is a framing $f: M^{2k+1} \to ESO_{2k+2}$ of M such that by a sequence of framed modifications of type $(k+1, k+1)$, the framed manifold (M^{2k+1}, f) may be reduced to the standard framed sphere (S^{2k+1}, g_0). Since M and $M \# \Sigma$ are diffeomorphic, it follows that $M \# \Sigma$ bounds a π-manifold W^{2k+2}. Let $F: W \to ESO_{2k+2}$ be a framing of the tangent bundle of W; then $f' = F \mid (M \# \Sigma)$ is a framing in the stable tangent bundle of $M \# \Sigma$.

We shall show that by a sequence of framed modifications the framed manifold $(M \# \Sigma, f')$ may be reduced to (Σ, g), where g is some framing of Σ. This follows from the preceding remarks if we perform certain modifications on the left-hand summand M of $M \# \Sigma$; the modifications are those that reduced (M, f) to (S^{2k+1}, g_0) We need only observe that these modifications of $M \# \Sigma$ may be framed with respect to *any* framing (in particular, with respect to the framing f'), since the obstructions (see [4]) to framing the modifications lie in the group $\pi_k(SO_{2k+1})$, which is zero for $k \equiv 2, 4, 6 \pmod 8$. It now follows that by pasting together W and the trace (that is, the framed cobordism between $M \# \Sigma$ and Σ) of these modifications leading from $(M \# \Sigma, f')$ to (Σ, g) by the identity map of $\partial W = M \# \Sigma$, we obtain a π-manifold bounded by Σ, as desired.

The considerably more complicated action of bP_n for $n = 4k - 1$ has been discussed by W. Browder [1]. Finally, if $k \equiv 6 \pmod 8$, then the assumption of almost parallelizability in Proposition 1 may be removed.

REFERENCES

1. W. Browder, *On the action of* $\Theta^n(\partial\pi)$, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton University Press, Princeton, 1965, pp. 23-36.

2. E. H. Brown, Jr. and F. P. Peterson, *The Kervaire invariant of* $(8k + 2)$*-manifolds*, Bull. Amer. Math. Soc. 71 (1965), 190-193.

3. E. H. Brown, Jr. and B. Steer, *A note on Stiefel manifolds*, Amer. J. Math. 87 (1965), 215-217.

4. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres: I*, Ann. of Math. (2) 77 (1963), 504-537.

5. A. Kosinski, *On the inertia group of* π*-manifolds*, Amer. J. Math. (to appear).

6. I. Tamura, *Classification des variétés différentiables*, $(n - 1)$*-connexes, sans torsion, de dimension* $2n + 1$, Séminaire Henri Cartan 15 (1962/63), Exp. 16 to 19.

7. A. Vasquez, *Structure of highly connected manifolds*, Thesis, University of California, Berkeley (1963).

Stanford University and University of California, Los Angeles