A NOTE ON KISTER'S ISOTOPY

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In [3] Kister gave a short proof of the following theorem: *Let* h *be a homeomorphism of* R^n *onto itself such that* $\|h - 1\|$ *is bounded. Then* h *is isotopic to* 1. Kister mentioned the resemblance of his method to Alexander's [1], and apparently Kister's method is slightly stronger, since his theorem immediately implies that of Alexander. The purpose of this note is to describe more clearly the relation between the two methods, and, *en passant*, to strengthen Kister's theorem.

Let B^n, $\dot{B}^n$, $\mathring{B}^n$ denote the unit ball of R^n, its boundary, and its interior, respectively. Let ϕ be any radial homeomorphism of $\mathring{B}^n$ onto R^n; that is, suppose

$$\frac{\phi(x)}{\|\phi(x)\|} = \frac{x}{\|x\|} \qquad (x \neq 0).$$

Call a homeomorphism h of R^n a *Kister homeomorphism* if $\|h - 1\|$ is bounded, and call a homeomorphism g of $\mathring{B}^n$ an *Alexander homeomorphism* if g extends to a homeomorphism of B^n which is the identity on $\dot{B}^n$.

THEOREM 1. h *is a Kister homeomorphism only if* $\phi^{-1} h \phi$ *is an Alexander homeomorphism.*

Proof. Let $b \in \dot{B}^n$. We wish to prove that $\lim_{x \to b} \phi^{-1} h \phi(x) = b$. Examine the inequality

$$\|\phi^{-1} h \phi(x) - b\| \leq \left\|\phi^{-1} h \phi(x) - \frac{\phi^{-1} h \phi(x)}{\|\phi^{-1} h \phi(x)\|}\right\| + \left\|\frac{\phi^{-1} h \phi(x)}{\|\phi^{-1} h \phi(x)\|} - \frac{x}{\|x\|}\right\| + \left\|\frac{x}{\|x\|} - b\right\|.$$

Evidently the first and last terms approach 0 as $x \to b$. Hence it suffices to prove that

$$\lim_{x \to b} \left\|\frac{\phi^{-1} h \phi(x)}{\|\phi^{-1} h \phi(x)\|} - \frac{x}{\|x\|}\right\| = 0. \tag{1}$$

By hypothesis, $\|h - 1\|$ is bounded, say by k. Hence $\|h\phi - \phi\| < k$. So, for $x \neq 0$,

$$\left\|\frac{h\phi(x)}{\|h\phi(x)\|} - \frac{\phi(x)}{\|h\phi(x)\|}\right\| < \frac{k}{\|h\phi(x)\|} \quad \text{and} \quad \left\|\frac{h\phi(x)}{\|\phi(x)\|} - \frac{\phi(x)}{\|\phi(x)\|}\right\| < \frac{k}{\|\phi(x)\|}.$$

Hence

$$\left\|\frac{h\phi(x)}{\|h\phi(x)\|} - \frac{\phi(x)}{\|\phi(x)\|}\right\| < k\left(\frac{1}{\|h\phi(x)\|} + \frac{1}{\|\phi(x)\|}\right).$$

Since ϕ^{-1} is radial, we get the inequality

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$$\left\| \frac{\phi^{-1} h \phi(x)}{\| \phi^{-1} h \phi(x) \|} - \frac{\phi^{-1} \phi(x)}{\| \phi^{-1} \phi(x) \|} \right\| < k \left(\frac{1}{\| h \phi(x) \|} + \frac{1}{\| \phi(x) \|} \right).$$

Now as $x \to b$, $\| h \phi(x) \| \to \infty$ and $\| \phi(x) \| \to \infty$, and we have established the truth of (1).

THEOREM 2. *Kister homeomorphisms are stable* (see [2] for the definition of the term "stable").

Proof. Choose the homeomorphism ϕ of Theorem 1 so that $\phi = 1$ on a neighbor hood of the origin. Then if h is a Kister homeomorphism, $\phi^{-1} h \phi$ is an Alexander homeomorphism. Extending by the identity map, we can view $\phi^{-1} h \phi$ as a stable homeomorphism of R^n that agrees with h on a neighborhood of the origin. Hence h is stable.

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