MAXIMAL IDEALS IN THE ALGEBRA OF BOUNDED HOLOMORPHIC FUNCTIONS

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1. INTRODUCTION

Let B(X) be the ring of bounded holomorphic functions on an open Riemann surface X, and assume that B(X) is a proper extension of the complex numbers $\mathfrak{C}$. With respect to the supremum norm, B(X) is a Banach algebra. (Because B(X) is semisimple, this is the only norm, up to equivalence.) Let $\mathcal{M}(X)$ be the maximal ideal space of B(X), endowed, as usual, with the weak-star topology. For $x \in X$, define

$$M(x) = \{f \in B(X) \mid f(x) = 0\}.$$

M(x) is a maximal ideal of B(X). We call such maximal ideals of *type* I; all others are called of *type* II.

In this paper, we obtain several characterizations of the ideals of type I. We must assume, however, that X is a relatively compact domain of another surface W, and that either the boundary of X in W consists of analytic simple closed curves or every boundary point is an essential singularity of some bounded holomorphic function on X.

2. PRELIMINARIES

Definition 1. Let W be any Riemann surface. By a *bounded* domain of W we mean a domain $X \subset W$ for which Cl X is compact and $\text{Cl } X \neq W$ (Cl X denotes the closure of X in W). A domain X is called a *finite* domain if it is bounded and the boundary of X in W equals the boundary of W - X and consists of a finite number of analytic simple closed curves.

If X is a domain of a compact Riemann surface W, and if the complement of X in W consists of a finite number of simply connected, nondegenerate continua, then it follows from the uniformization theory for Riemann surfaces that X is conformally equivalent to a finite Riemann surface. Our results for finite surfaces remain valid for this kind of surface.

LEMMA 1. *Let* X *be a bounded domain of a Riemann surface* W, *and let* $x \in \text{Cl } X$. *Then there exists an* $f \in B(X)$ *such that*

(1) f *is analytic in a neighborhood of* Cl X,

(2) f *has a simple zero at* x *and no other zeros, and*

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(3) f *is bounded away from zero outside every open neighborhood of* x.

Proof. Since Cl X $\neq$ W, there exists a $w \in W$ such that Cl X $\subset W - w = X_1$. Because X_1 is an open Riemann surface, the generalized Weierstrass theorem (see, for example, Behnke and Sommer [4, p. 591]) implies that there exists a holomorphic function f on X_1 that vanishes of order 1 at x and has no other zeros. This is the required function.

If Y is a topological space, denote by C(Y) the ring of continuous complex-valued functions on Y. Let X be a Riemann surface. Then every $f \in B(X)$ has an extension $f^\wedge \in C(\mathcal{M}(X))$. This extended function is defined as follows:

$$f^\wedge(M) = \alpha \Leftrightarrow f - \alpha \in M, \quad M \in \mathcal{M}(X).$$

By the theorem of Gelfand and Mazur, there exists a one-to-one correspondence between the maximal ideals of a commutative Banach algebra with identity and the $\mathfrak{C}$-homomorphisms of the algebra onto $\mathfrak{C}$. Thus $f^\wedge$ is well-defined. Henceforth, the symbol M will stand for both a maximal ideal and the corresponding homomorphism. (Thus $f^\wedge(M) = M(f)$.) If $M = M(x)$ for some $x \in X$, then $f^\wedge(M) = f(x)$.

For the case where X is a bounded domain of the Riemann surface W, we introduce some more terminology:

$$B_c(X) = \{f \in B(X) \mid f \text{ has a continuous extension to Cl } X\}$$
$$= \{f \in C(\text{Cl } X) \mid f \text{ is holomorphic on } X\},$$
$$\mathcal{M}_c(X) = \text{maximal ideal space of } B_c(X),$$

and for $x \in$ Cl X

$$M_c(x) = \{f \in B_c(X) \mid f(x) = 0\}.$$

LEMMA 2. *Let* X *be a bounded domain. Then for* $x \in X$, *the maximal ideals* M(x) *and* $M_c(x)$ *of* B(X) *and* $B_c(X)$, *respectively, are principal. Moreover, these ideals have generators that are analytic on* Cl X.

Proof. Let f be the function described in the previous lemma. Then $f \in M_c(x) \subset M(x)$, and f is analytic on Cl X. Let g belong to M(x) (to $M_c(x)$, respectively). Then clearly g/f is holomorphic on X. Choose a relatively compact neighborhood U of x such that Cl U $\subset$ X. Then f is bounded away from zero on Cl X - U. Thus g/f is bounded in X - U. Also, g/f is bounded on U. Thus g/f is bounded on X. If g is continuous on Cl X, then g/f is also continuous on Cl X.

Let X be a bounded domain. It is well known that the map ϕ that sends $x \in$ Cl X into $M_c(x) \in \mathcal{M}_c(X)$ is a homeomorphism. (That ϕ is surjective was shown by Arens [3].) Similarly, for every Riemann surface X there exists a natural continuous map ϕ that sends each point $x \in X$ into $M(x) \in \mathcal{M}(X)$. Whenever X is a bounded domain, ϕ is injective, and we call ϕ the *natural injection* of X into $\mathcal{M}(X)$.

LEMMA 3. *Let* X *be a bounded domain of a Riemann surface* W. *Then there exists a continuous map* $\psi: \mathcal{M}(X) \to$ Cl X *such that if* ϕ *is the natural injection of* X *into* $\mathcal{M}(X)$, *then*

$$\psi\phi = \textit{Identity on } X \quad \textit{and} \quad \phi\psi = \textit{Identity on } \phi(X).$$

In particular, X *is homeomorphic to* $\phi(X)$.

Proof. Let $M \in \mathcal{M}(X)$. Then M is a $\mathfrak{C}$-homomorphism of B(X). The mapping $M \mid B_c(X)$ is also a $\mathfrak{C}$-homomorphism, and is thus determined by a unique point $\psi(x) \in \mathrm{Cl}\, X$, by the result of Arens [3]. The topologies of $\mathcal{M}(X)$ and Cl X are both weak-star topologies. Thus ψ is continuous. The rest of the lemma is obvious.

Definition 2. Let X be a bounded domain of the Riemann surface W. Let $x \in \mathrm{Cl}\, X$, and define the *fiber of* $\mathcal{M}(X)$ *over* x to be

$$\mathcal{M}_x = \{M \in \mathcal{M}(X) \mid \psi(M) = x\},$$

where ψ is the map of the previous lemma.

Remark. The fibers of Definition 2 are not new. They appear in Hoffman's book [6, p. 161] as well as in the work of Alling [2].

LEMMA 4. *Let* X *be a bounded domain of a Riemann surface* W. *Let* $x \in X$ *and* $M \in \mathcal{M}_x$. *Then* $M = M(x)$.

Proof. $M \cap B_c(X) = \{g \in B_c(X) \mid g(x) = 0\}$. Let f be the function described in Lemma 1. Then $f \in M \cap B_c(X)$, and $(f) = fB(X) \subset M$. By Lemma 2, (f) is maximal. Thus $M(x) = (f) = M$.

COROLLARY. *Let* X *be a bounded domain of a Riemann surface* W. *Then* X *is homeomorphic to an open subset of* $\mathcal{M}(X)$.

Proof. Using the notation of Lemma 3, we see that $\phi(X) = \psi^{-1}(X)$. The set X is open in Cl X, and ψ is continuous. Thus $\phi(X)$ is open in $\mathcal{M}(X)$.

LEMMA 5. *Let* X *be a finite domain of the Riemann surface* W. *Then for each discrete sequence* $\{x_n\} \subset X$, *there exists an* $f \in B(X)$ *such that* $\lim_{n \to \infty} f(x_n)$ *does not exist.*

Proof. Ahlfors [1] has shown that there exists a mapping p, analytic in a neighborhood of Cl X, that is an N-to-one covering of the closed unit disc, for some positive integer N. Moreover, $p \mid X$ is an N-to-one covering of the interior of the closed unit disc, and $p \mid \mathrm{Cl}\, X - X$ is an N-to-one covering of the unit circle. Because Cl X is compact, we may assume (by choosing a subsequence) that $x_n \to x \in \mathrm{Cl}\, X - X$. Then $p(x_n) \to 1$ and $|p(x_n)| < 1$. Again, we may choose a subsequence such that $p(x_n)$ is distinct and infinite and constitutes an interpolating sequence (see Hoffman [6, pp. 194-204]). Choose a bounded analytic function f on the unit disc such that $f(p(x_{2n+1})) = 0$ and $f(p(x_{2n+2})) = 1$ for $n = 0, 1, 2, \cdots$. Then $f \circ p \in B(X)$, and $\lim_{n \to \infty} (f \circ p)(x_n)$ does not exist.

3. CHARACTERIZATION OF THE IDEALS OF TYPE I

THEOREM 1. *Let* X *be a finite domain of the Riemann surface* W. *Then the following are equivalent for* $M \in \mathcal{M}(X)$.

(1) M *is of type* I (*that is, there exists an* $x \in X$ *such that* $M = M(x)$).

(2) $\mathcal{M}(X)$ *satisfies the first countability axiom at* M.

(3) M *has a neighborhood in* $\mathcal{M}(X)$ *that is homeomorphic to the open unit disc.*

(4) M *is a principal ideal.*

(5) $B(X)(M \cap B_c(X))$ *is a maximal ideal in* B(X).

(6) $B(X)(M \cap B_c(X)) = M$.

(7) *No element of* M *possesses roots of all orders in* B(X).

Proof. We show that (1) $\Leftrightarrow$ (4), (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (1), and (1) $\Leftrightarrow$ (7).

That (1) $\Rightarrow$ (4) is the content of Lemma 2.

(4) $\Rightarrow$ (1). Assume $M = (f)$. Since f is not a unit, $\inf\{|f(x)| \mid x \in X\} = 0$. Thus there exist $x_n \in X$ such that $\lim_{n\to\infty} f(x_n) = 0$. Let $g \in M$; then there is an $h \in B(X)$ such that $g = fh$. Because h is bounded, $\lim_{n\to\infty} g(x_n) = 0$. The ideal (f) is maximal. Hence, for each $g \in B(X)$, there is a $\lambda \in \mathfrak{C}$ such that $g - \lambda \in (f)$. It follows that $\lim_{n\to\infty} g(x_n)$ exists for all $g \in B(X)$. Furthermore, if $g \in B(X)$, then $g \in M$ if and only if $\lim_{n\to\infty} g(x_n) = 0$. By Lemma 5, $\{x_n\}$ has no discrete subsequences. We may assume that $x_n \to x \in X$. Thus, if $g \in B(X)$, then $g \in M$ if and only if $g(x) = 0$. Thus $M = M(x)$.

(3) $\Rightarrow$ (2). This is trivial.

(2) $\Rightarrow$ (1). Since $\mathscr{M}(X)$ satisfies the first countability axiom at M, the topology at M can be described in terms of sequences. Alling [2] has shown that $\phi(X)$ is dense in $\mathscr{M}(X)$, where ϕ is the natural injection of X into $\mathscr{M}(X)$. Thus there exists a sequence $\{x_n\} \subset X$ with $M(x_n) \to M$. Thus $f(x_n) \to \hat{f}(M)$ for all $f \in B(X)$. Lemma 5 implies that we may choose a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that $x_n \to x \in X$. By continuity of the map ϕ, $M(x_n) \to M(x)$. Thus $M(x) = M$, because $\mathscr{M}(X)$ is a Hausdorff space.

(1) $\Rightarrow$ (3). This is an immediate consequence of the corollary to Lemma 4.

(1) $\Rightarrow$ (6). By Lemma 2, the generator of the principal ideal M may be chosen in $B_c(X)$.

(6) $\Rightarrow$ (5). This is trivial.

(5) $\Rightarrow$ (1). $M \cap B_c(X) = M_c(x)$ for some $x \in \mathrm{Cl}\, X$. If $x \in X$, then by Lemma 4, $M = M(x)$. Assume that $x \in \mathrm{Cl}\, X - X$. Choose $\{x_n\} \subset X$ such that $x_n \to x$. Let $f \in B(X)M_c(x)$. Then $f = \sum_{i=1}^n f_i g_i$ with $f_i \in B(X)$ and $g_i \in M_c(x)$. Hence $\lim_{n\to\infty} g_i(x_n) = 0$. The f_i are bounded. Thus $\lim_{n\to\infty} f(x_n) = 0$. Because $B(X)M_c(x)$ is maximal in B(X), there exists for each $f \in B(X)$ a $\lambda \in \mathfrak{C}$ such that $f - \lambda \in B(X)M_c(x)$. Thus $\lim_{n\to\infty} f(x_n)$ exists for all $f \in B(X)$. This contradicts Lemma 5.

(1) $\Rightarrow$ (7). If $f \in M$, then f has a zero of order $n \geq 1$ at $x \in X$. Thus f does not have an $(n+1)$st root in B(X).

(7) $\Rightarrow$ (1). Assume that $M \cap B_c(X) = M_c(x)$ and $x \in \mathrm{Cl}\, X - X$. Let p be any Ahlfors map (see the proof of Lemma 5). We may assume that $p(x) = 1$. Let $g(z) = z - 1$ for $z \in \mathrm{Cl}\, D$, where $D = \{z \in \mathfrak{C} \mid |z| < 1\}$. Then $f = g \circ p \in M_c(x)$. For each positive integer n, there exists a $g_n \in B_c(D)$ such that $g_n^n = g$. Now $f_n = g_n \circ p \in M_c(x)$, and $f_n^n = f$. This contradicts the hypothesis.

Remarks. (1) The equivalence of conditions (1) to (4) was conjectured by Kakutani [7]. He proved that (1) and (4) are equivalent for finite domains of the complex sphere.

(2) Gleason [5] has shown that in any commutative Banach algebra with identity, (4) implies (3) unless the principal maximal ideal is isolated in the maximal ideal space of the algebra. Under the hypothesis of the theorem, M cannot be isolated. It is well-known (see for example Rickart [9, p. 168]) that the maximal ideal space of an algebra without idempotents is connected.

THEOREM 2. *Let* X *be a bounded domain of a Riemann surface* W. *Assume that every boundary point of* X *is an essential singularity for some bounded holomorphic function on* X. *Let* $M \in \mathcal{M}(X)$. *Then* M *is of type* I *if and only if* M *is a principal ideal and* $\bigcap_{n=1}^{\infty} M^n = \{0\}$.

Proof. If M is of type I, then by Lemma 2, M is a principal ideal. Let $M = M(x)$, with $x \in X$. If $f \in M^n$, then f has a zero of order at least n at x. Thus $\bigcap_{n=1}^{\infty} M^n = \{0\}$.

To prove the converse, note that $M \cap B_c(X) = \{f \in B_c(X) \mid f(x) = 0\}$ for a unique $x \in \mathrm{Cl}\, X$. By Lemma 4, it suffices to show that $x \in X$. Let $f \in B(X)$ generate the principal maximal ideal M. $\mathcal{M}(X)$ is connected (Rickart [9, p. 168]). Thus (Gleason [5]) there is a neighborhood U of the origin in $\mathfrak{C}$ such that if $\phi: N \to N(f)$ for $N \in \mathcal{M}(X)$, then ϕ is a homeomorphism of $\phi^{-1}(U)$ onto U. Shrinking U, if necessary, we now see (either from [5] or from [8]), that for each $g \in B(X)$, $g^\wedge$ can be expanded in a power series

$$g^\wedge(N) = \sum_{n=0}^{\infty} a_n f^\wedge(N)^n \qquad (a_n \in \mathfrak{C})$$

that converges in $\phi^{-1}(U)$. We may without loss of generality assume $U = \{z \in \mathfrak{C} \mid |z| < \varepsilon\}$ for some $\varepsilon > 0$. Now, if $a_n = 0$ for all n, then $g \in M^n$ for all n; that is, $g = 0$. Let g_0 be the function of Lemma 1; then g_0 is analytic in a neighborhood of Cl X, vanishes of order 1 at x, and is bounded away from zero outside every neighborhood of x. Clearly $g_0 \in M \cap B_c(X)$, and if

$$g_0^\wedge(N) = \sum_{n=0}^{\infty} b_n f^\wedge(N)^n \qquad \text{in } \phi^{-1}(U),$$

then $b_0 = 0$.

There exists a smallest k such that $b_k \neq 0$. The function $f^\wedge \mid \phi^{-1}(U)$ covers U precisely once, and $g_0^\wedge$ is an analytic function of $f^\wedge$ in $\phi^{-1}(U)$. Thus there exists a $\delta > 0$ such that each $\beta \in \mathfrak{C}$ with $0 < |\beta| < \delta$ is covered exactly k times by $g_0^\wedge \mid \phi^{-1}(U)$. Let ψ be the map described in Lemma 3; then $g_0(\psi(N)) = g_0^\wedge(N)$ for $N \in \mathcal{M}(X)$, because $\psi(N)$ is the homomorphism N restricted to $B_c(X)$ and $g_0 \in B_c(X)$. Choose δ small enough so that for $|\beta| < \delta$ the equation $g_0(y) = \beta$ has at most one solution $y \in X$, and choose an $x_1 \in X$ such that $g_0(x_1) = \beta$ with $0 < |\beta| < \delta$; then there exists a set $\{N_0, \cdots, N_{k-1}\} \subset \phi^{-1}(U)$ such that

$$\psi(N_0) = \cdots = \psi(N_{k-1}) = x_1.$$

By Lemma 4, $N_0 = \cdots = N_{k-1} = M(x_1)$. Thus $k = 1$, and $f^\wedge$ is a holomorphic function of $g_0^\wedge$ in $\phi^{-1}(U)$. Hence, for all $g \in B(X)$, $g^\wedge$ is a holomorphic function of $g_0^\wedge$ in $\phi^{-1}(U)$. Clearly, this means that every $g \in B(X)$ can be expanded in a convergent power series in g_0, in some neighborhood of x in W. This is only possible if $x \in X$, because every boundary point is an essential singularity for some bounded holomorphic function on X.

Remark. Theorem 2 is a generalization of Theorem 1, because every boundary point of a finite domain is an essential singularity for some bounded holomorphic function. The unit disc certainly has this property. The general case is reduced to the case of the unit disc *via* any Ahlfors map. (See the proof of Lemma 5.)

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