SIMULTANEOUS INTERPOLATION IN H^2

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1. INTRODUCTION

In 1961, H. Shapiro and A. Shields [4] investigated weighted interpolation in several function spaces. Our work is an extension of the part of their paper that deals with the Hardy space H^2. The author acknowledges gratefully many suggestions communicated orally by Professors Shapiro and Shields.

Let $\{z_n\}$ denote a fixed sequence of complex numbers such that $0 \leq |z_n| \leq |z_{n+1}| < 1$, and let L_n and L_n' be the functionals defined on H^2 by $L_n f = f(z_n)$ and $L_n' f = f'(z_n)$ ($f \in H^2$, $n = 1, 2, \cdots$). In [4] it is shown that if $|z_n|$ tends to 1 rapidly enough, then, for each sequence $\{w_n\}$ in ℓ_2, there exists a corresponding f in H^2 satisfying the weighted interpolation condition

$$(1) \qquad f(z_n) = w_n \|L_n\| .$$

Our main result is that under a slightly stronger restriction on $\{z_n\}$, it follows that for every pair of sequences $\{w_n\}$ and $\{w_n'\}$ in ℓ_2, there exists a corresponding f in H^2 that satisfies the *simultaneous* interpolation conditions

$$(2) \qquad f(z_n) = w_n \|L_n\|, \qquad f'(z_n) = w_n' \|L_n'\| .$$

2. NOTATION AND BACKGROUND MATERIAL

For each f analytic in the open unit disc,

$$\lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$$

exists (possibly as infinity) and equals $\sum |a_n|^2$, where the a_n are the Maclaurin coefficients of f. The Hardy space H^2 consists of the functions f for which the limit is finite. If $f \in H^2$, the radial limit $f(e^{i\theta})$ exists almost everywhere and belongs to the Lebesgue space $\mathscr{L}^2[0, 2\pi]$.

H^2 becomes a Hilbert space if we define the inner product of any two of its members f and g to be $\sum a_n \bar{b}_n$, where the a_n and b_n are the Maclaurin coefficients of f and g, respectively. The mapping $f(re^{i\theta}) \to f(e^{i\theta})$ is an isometry from H^2 into $\mathscr{L}^2$.

Let $K_n(z)$ and $K_n'(z)$ denote $(1 - \bar{z}_n z)^{-1}$ and $z(1 - \bar{z}_n z)^{-2}$, respectively (note that K_n' is not the derivative of K_n). It is easily verified that these functions belong to H^2 and that

$$(f, K_n) = L_n f = f(z_n), \quad (f, K_n') = L_n' f = f'(z_n) \quad (n = 1, 2, \cdots)$$

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for all f in H^2. It follows from the above that

$$\|L_n\| = \|K_n\| = \delta_n \quad \text{and} \quad \|L_n'\| = \|K_n'\| = \delta_n^3(1 + |z_n|^2)^{1/2},$$

where δ_n denotes $(1 - |z_n|^2)^{-1/2}$. The equations (2) now take the form

$$(f, K_n/\|K_n\|) = w_n, \quad (f, K_n'/\|K_n'\|) = w_n'.$$

Some basic facts about H^2 can be found in [5] under the heading H^p $(p \geq 1)$.

If $A = (a_{ij})$ is an $N \times N$ Hermitian matrix and M is the greatest of the N numbers $\sum_j |a_{ij}|$ $(1 \leq i \leq N)$, then the norm of A, as an operator on the Hilbert space E_N, is not greater than M. This follows from Theorem 3.3 in [2] and from the fact that $\|A\| = \max_k |\lambda_k|$ (λ_k in the spectrum of A).

We shall say that $\{z_n\}$ *approaches the boundary exponentially* if

$$(E_\sigma) \qquad \delta_n/\delta_{n+1} \leq \sigma < 1 \qquad (n = 1, 2, \cdots)$$

for some σ. We shall say that the sequence is a *Carleson sequence* if

$$(C_\delta) \qquad \prod_{k \neq n} \left|\frac{z_k - z_n}{1 - \bar{z}_n z_k}\right| > \delta > 0 \qquad (n = 1, 2, \cdots)$$

for some δ.

3. PRELIMINARY RESULTS

In this section we shall establish the relations

$$(E_\sigma) \Rightarrow (C_\delta) \Rightarrow (1) \text{ is solvable},$$

the implications being equivalences if the z_n all lie on a radius.

THEOREM 1. *If* $\{z_n\}$ *satisfies* (E_σ) *for some* σ, *it satisfies* (C_δ) *for some* δ.

The first two theorems in [3] imply this result. The following is a direct proof.

Proof. The identity

$$(3) \qquad \left|\frac{a - b}{1 - \bar{b}a}\right|^2 = 1 - \frac{(1 - |a|^2)(1 - |b|^2)}{|1 - \bar{b}a|^2} \qquad (|a|, |b| < 1)$$

implies that

$$\left|\frac{a - b}{1 - \bar{b}a}\right| \geq \frac{||a| - |b||}{1 - |ba|} \qquad (|a|, |b| < 1).$$

If x_n denotes $(1 - |z_n|)^{-1}$ and if $i < n$, it follows that

$$\left|\frac{z_i - z_n}{1 - \bar{z}_n z_i}\right| \geq \frac{|x_n - x_i|}{x_i + x_n - 1} > \frac{x_n - x_i}{x_n + x_i} = 1 - \frac{2x_i}{x_n + x_i} \geq 1 - \frac{2}{1 + \sigma^{-2|i-n|}}.$$

Since the extremes are unaffected if i and n are interchanged, the inequality holds for all i and n with $i \neq n$. Thus, for each n,

$$\prod_{i\neq n}\left|\frac{z_i - z_n}{1 - \bar{z}_n z_i}\right| \geq \prod_{i\neq n}\left[1 - \frac{2}{1+\sigma^{-2|i-n|}}\right]$$

$$= \left\{\prod_{k=1}^{n-1}\left[1 - \frac{2}{1+\sigma^{-2k}}\right]\right\}\left\{\prod_{k=1}^{\infty}\left[1 - \frac{2}{1+\sigma^{-2k}}\right]\right\} \geq \prod_{k=1}^{\infty}\left[1 - \frac{2}{1+\sigma^{-2k}}\right]^2.$$

This product is positive, since $\sum_k (1+\sigma^{-2k})^{-1} < \infty$; therefore we may take δ to be its value. This proves that $\{z_n\}$ is a Carleson sequence. ■

Shapiro and Shields [4] showed that if (C_δ) holds, then (1) is solvable. We include a sketch of their proof, since we use the elements of the proof in the next section.

THEOREM 2. *If* $\{z_n\}$ *is a Carleson sequence, then for every sequence* $\{w_n\}$ *in* ℓ_2 *there exists an* f *in* H^2 *satisfying the interpolation condition* (1).

Outline of proof. Let $\{w_n\}$ be a fixed sequence in ℓ_2. For $1 \leq n \leq N$, define

$$B_{Nn}(z) \equiv \prod_k \frac{z - z_k}{1 - \bar{z}_k z} \quad (1 \leq k \leq N,\ k \neq n), \qquad b_{Nn} \equiv B_{Nn}(z_n),$$

$$\phi_{Nn}(z) \equiv B_{Nn}^2(z)/(1 - \bar{z}_n z)^2 b_{Nn}^2 \delta_n^3, \qquad f_N \equiv \sum_1^N w_n \phi_{Nn}.$$

We see that $f_N(z_n) = w_n \delta_n$ $(1 \leq n \leq N)$. It is shown in [3] that $\{\|f_N\|\}$ is bounded, from which it follows that $\{f_N\}$ has a weak cluster point, say f. Then, for each n, $\{(f_N, K_n)\}$ has the cluster point (f, K_n). But $(f_N, K_n) = w_n \delta_n$ for $N \geq n$, so that (f, K_n) must equal $w_n \delta_n$. ■

THEOREM 3. *If* $\{z_n\}$ *is a radial sequence, then the solvability of* (1) *is equivalent to exponential approach to the boundary.*

This theorem follows from the identity (3) and the following three lemmas.

LEMMA 1. *If, for each sequence* $\{w_n\}$ *in* ℓ_2, (1) *is satisfied by some* f *in* H^2, *then there exists a constant* M *such that for each sequence* $\{w_n\}$ *in the unit ball of* ℓ_2, (1) *is satisfied by an* f *in* H^2 *whose norm does not exceed* M.

LEMMA 2. *Suppose* M *is a constant, and suppose that for each pair* i, j *of indices* $(i \neq j)$, *a function* f_{ij} *in* H^2 *exists such that* $\|f_{ij}\| \leq M$ *and*

$$(4) \qquad (f_{ij}, K_i/\|K_i\|) = 0, \qquad (f_{ij}, K_j/\|K_j\|) = 1.$$

Let

$$\xi_{ij} \equiv |(1 - \bar{z}_i z_j)|^{-1} \delta_i^{-1} \delta_j^{-1} = \left(\frac{K_i}{\|K_i\|}, \frac{K_j}{\|K_j\|}\right).$$

Then $|\xi_{ij} - 1| > 1/M^2$.

A sequence is said to be *weakly separated* if there exists a constant δ for which

$$\left|\frac{z_i - z_j}{1 - \bar{z}_j z_i}\right| \geq \delta > 0 \qquad (i \neq j).$$

LEMMA 3. *If a weakly separated sequence lies on a radius, it approaches the boundary exponentially.*

The first and third lemmas are proved in [4] (see p. 517 and p. 529, respectively). A proof of half of the equivalence in Lemma 2, ostensibly for two particular Hilbert spaces (neither of them H^2), but actually valid for any Hilbert function space, is given in [4]. We now present a somewhat simpler proof.

Proof of Lemma 2. For $i \neq j$, the function of minimal norm satisfying (3) clearly has the form

$$f_{ij} = c_i K_i + c_j K_j. \tag{5}$$

The values c_i and c_j are readily found from the relations $(f_{ij}, K_i) = 0$ and $(f_{ij}, K_j) = \|K_j\|$. Substituting the results into (5) and then taking the inner product of the members of (5) with themselves, we find that $\|f_{ij}\|^2 = (1 - \xi_{ij})^{-1}$. ■

4. THE MAIN RESULT

In the last section we showed that (E_σ) implies the solvability of (1), the implication becoming an equivalence if $\{z_n\}$ is radial. In this section, we shall prove that (E_σ) implies the solvability of (2).

THEOREM 4. *If* $\{z_n\}$ *tends to the boundary exponentially and if* $\{w_n\}$ *and* $\{w_n'\}$ *are sequences in* ℓ_2, *then there exists an* f *in* H^2 *satisfying the simultaneous interpolation conditions* (2).

Proof. We shall find two functions F and G in H^2 such that

$$F(z_n) = w_n \delta_n, \qquad F'(z_n) = 0, \tag{6}$$

$$G(z_n) = 0, \qquad G'(z_n) = w_n' \delta_n^3 (1 + |z_n|^2)^{1/2}. \tag{7}$$

Then $F + G$ can serve as the desired function.

The determination of F. We shall construct a sequence $\{F_N\}$ in H^2 such that $\{\|F_N\|\}$ is bounded and F_N satisfies (6) for $1 \leq n \leq N$. $\{F_N\}$ will then have a weak cluster point that can be taken as F.

For $1 \leq n \leq N$, let B_{Nn}, b_{Nn}, and f_N be defined as in Theorem 2. Let

$$b_{Nn}' \equiv B_{Nn}'(z_n) = b_{Nn} \sum_{k=1}^{n} \delta_k^{-2} (z_n - z_k)^{-1} (1 - \bar{z}_k z_n)^{-1} \qquad (k \neq n),$$

and let

$$F_{Nn} \equiv \alpha_{Nn}(z - z_n)(1 - \bar{z}_n z)^{-3} B_{Nn}^2(z),$$

where

$$\alpha_{Nn} \equiv -2 b_{Nn}^{-2} \delta_n^{-3} (\bar{z}_n + b_{Nn}' \delta_n^{-2} b_{Nn}^{-1}).$$

Finally, let $F_N \equiv f_N + \sum_1^N w_n F_{Nn}$.

Simple computations show that F_N satisfies (6) for $1 \leq n \leq N$. Since $\|F_N\| \leq \|f_N\| + \|\sum_1^N w_n F_{Nn}\|$, it suffices to show that $\{\|\sum_1^N w_n F_{Nn}\|\}$ is bounded ($N = 1, 2, \cdots$). To do this, we shall show that as operators on E_N the matrices $A_N = ((F_{Ni}, F_{Nj}))$ $(1 \leq i, j \leq N)$ have uniformly bounded norms.

The Carleson condition is stronger than weak separation, which in turn is equivalent (see [4, p. 529]) to the existence of a constant $c > 0$ such that

$$(8) \qquad |z_i - z_j|\, \delta_k \geq c \qquad \text{for } k = i \text{ and for } k = j \ (i \neq j).$$

Since $|1 - \bar{z}_i z_j|\, \delta_k^2 \geq 1/2$ $(k = i, j)$, it follows that the terms on the right side of

$$b'_{Nn}/b_{Nn}\, \delta_n = \sum_k 1/\delta_k^2\, \delta_n^2\, (z_n - z_k)(1 - \bar{z}_k z_n)$$

are dominated by both δ_n^2/δ_k^2 and δ_k^2/δ_n^2. Thus they are dominated by $\sigma^{2|k-n|}$, so that the sum is dominated by $2\sigma^2\ (1 - \sigma)$. Together with the Carleson condition, this implies that α_{Nn} is dominated by δ_n^{-3}; therefore

$$|(F_{Ni}, F_{Nj})| \leq \frac{M}{\delta_i^3\, \delta_j^3} \left| \int_{-\pi}^{\pi} \frac{d\theta}{(z - z_j)(1 - \bar{z}_j z)(\bar{z} - \bar{z}_i)(1 - z_i \bar{z})} \right|$$

$$(z = e^{i\theta},\ 1 \leq i, j \leq N,\ i \neq j),$$

for some M that is independent of N. The integral can be evaluated by residues; it has the value

$$\frac{\delta_j^2\, z_j}{(z_j - z_i)(1 - \bar{z}_i z_j)} + \frac{\delta_i^2\, z_i}{(z_i - z_j)(1 - \bar{z}_j z_i)}.$$

By (8), each term in this expression is dominated by $\delta_j^2\, \delta_i^2 / |1 - \bar{z}_j z_i|$; therefore

$$|(F_{Ni}, F_{Nj})| \leq M'/\delta_i\, \delta_j\, |1 - \bar{z}_j z_i|,$$

for some constant M' independent of N. A similar computation shows that this is also the case if $i = j$. Thus, the terms of our matrix exceed neither $2M'\delta_i/\delta_j$ nor $2M'\delta_j/\delta_i$; therefore the ℓ_1-norms of the rows of A_N do not exceed $2M'(1 + 2\sigma/(1 - \sigma))$. Thus, the existence of a function F with the desired properties is assured.

The determination of G. The condition (E_σ) implies that the terms of $\sum (1 - |z_n|)$ are dominated by σ^n; therefore, the formal Blaschke product $B(z)$ formed from the sequence $\{z_n\}$ converges.

Let $b'_n \equiv B'(z_n)$. A computation shows that

$$b'_n = \bar{z}_n z_n^{-1}\, \delta_n^2 \prod_{k \neq n} \bar{z}_k z_k^{-1} (z_n - z_k)(1 - \bar{z}_k z_n)^{-1}.$$

Since $\{z_n\}$ is a Carleson sequence, δ_n^2/b'_n is bounded, and therefore

$$\{w'_n (1 + |z_n|^2)^{1/2}\, \delta_n^2\, b_n'^{-1}\}$$

is in ℓ_2. Thus, by Theorem 2, we can find a g in H^2 such that

$$g(z_n) = w'_n(1 + |z_n|^2)^{1/2} \delta_n^3 b_n'^{-1}.$$

Let $G \equiv Bg$. Since $|B(z)| \leq 1$ $(|z| < 1)$, it follows that G is in H^2. It is easily seen that G satisfies (7). This completes the proof of the theorem.

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