ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF $x'' + a(t)x = 0$

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The differential equation

$$x'' + a(t)x = 0 \qquad (t \geq 0) \tag{1}$$

has been widely investigated; see, for example, Cesari [2, pp. 80-90] and Bellman [1, Part 3]. In the special case when $a(t) > 0$ and

$$a(t) \uparrow \infty \quad \text{as } t \uparrow \infty, \tag{2}$$

Milloux [6], Hartman [4], and Prodi [7] have shown that (1) must have at least one nontrivial solution $x(t)$ such that

$$x(t) \to 0 \quad \text{as } t \to \infty. \tag{3}$$

Milloux [6] and Galbraith, McShane, and Parrish [3] have shown that in order that (3) holds for all solutions of (1), some additional condition on $a(t)$ besides (2) is needed. Furthermore, such a condition cannot be a simple restriction on the order of $a(t)$ or $a'(t)$, for Willett [9] has recently shown that for each $b(t) \geq 0$, there exists a function $a(t)$ such that $a'(t) \geq b(t)$ and (1) has a solution $x(t)$ with $\limsup_{t\to\infty} |x(t)| > 0$.

Sufficient conditions assuring that (3) holds for all solutions of (1) have been obtained (see, for example, [1, p. 88], [2, pp. 85-86], and [5]). One of the best of these conditions from a theoretical viewpoint is due to Sansone [8]: if $a(t)$ belongs to the class $C^1[0, \infty]$, then for every sequence $\{t_n\}$ subject to the conditions

$$t_n \to \infty, \qquad t_{n+1} - t_n \leq t_n - t_{n-1}, \qquad \limsup_{n\to\infty} \frac{t_{n+1} - t_n}{t_n - t_{n-1}} = 1,$$

it is true that

$$\sum_{n=1}^{\infty} (t_{n+1} - t_n) \min_{t_n \leq t \leq t_{n+1}} \frac{a'(t)}{a(t)} = \infty. \tag{4}$$

One of the reasons for the present paper is that for a given $a(t)$, conditions such as (4) are usually difficult to verify. It is our aim to present conditions on $a(t)$ that are usually easy to verify in practice, that imply that all solutions of (1) satisfy (3), and that in our opinion are quite general—for example, as general as Sansone's condition mentioned above. In its most general form, our condition requires the choice of a second function $q(t)$ that is in some sense a "smooth approximation" of $a(t)$. In many cases, the choice of $q(t)$ is obvious, and thus the condition is easily verified.

Let $\max(w(t), 0)$ be denoted by $w_+(t)$. Throughout the paper, we assume that $a(t)$ is of class $C^1[0, \infty)$ and that $a > 0$, $a' \geq 0$, and $a(t) \to \infty$ as $t \to \infty$.

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THEOREM. *If there exists a function* $q \in C^2[0, \infty)$ *such that* $q > 0$, $q' \geq 0$, $q(t) \to \infty$ *as* $t \to +\infty$,

(5) $$\lambda = \limsup_{t \to +\infty} \frac{1}{q(t)} \int_0^t \frac{|q''(\tau)|}{a^{1/2}(\tau)} d\tau < \frac{1}{2},$$

and

(6) $$\lim_{t \to +\infty} \frac{1}{q(t)} \int_0^t \left(2\gamma \frac{q'}{q} - \frac{a'}{a}\right)_+ q\, d\tau = 0 \quad \textit{for some constant } \gamma > 0,$$

then each solution of (1) *satisfies the condition*

(7) $$x(t) \to 0 \quad \textit{and} \quad a^{-1/2}(t)x'(t) \to 0 \quad \textit{as } t \to +\infty.$$

COROLLARY 1. *If there exists a positive* $p(t) \in C^1[0, \infty)$ *such that*

(8) $$\int_0^{+\infty} p^{-1}(t)\, dt = +\infty,$$

(9) $$\liminf_{t \to +\infty} \frac{p'(t)}{p(t)a^{1/2}(t)} \geq 0,$$

and

(10) $$\liminf_{t \to +\infty} \frac{a'(t)p(t)}{a(t)} > 0,$$

then (7) *holds for all solutions of* (1).

COROLLARY 2 (Sansone [8]). *If*

(11) $$\liminf_{t \to \infty} a'(t) > 0 \quad \textit{and} \quad \int_0^{+\infty} a^{-1}(t)\, dt = +\infty,$$

then (7) *holds for all solutions of* (1).

In particular, Corollary 2 holds if $a'(t) \to +\infty$ and the second requirement in (11) is satisfied. This is one of Sansone's sufficient conditions reported in [2, pp. 85-86], modulo the correction of a misprint.

COROLLARY 3. *If*

(12) $$\liminf_{t \to \infty} \frac{t\, a'(t) \log t}{a(t)} > 0,$$

then (7) *holds for all solutions of* (1).

COROLLARY 4. *If* $a(t) \in C^2[0, \infty)$, $a' > 0$, *and*

(13) $$\limsup_{t \to \infty} \frac{a''(t)}{a^{1/2}(t)a'(t)} \leq 0,$$

then (7) *holds for all solutions of* (1).

COROLLARY 5. *If* $a(t) \in C^2[0, \infty)$ *and*

$$ta''(t) \geq -a'(t) \quad \textit{for all } t \geq t_0, \tag{14}$$

for some constant t_0, *then* (7) *holds for all solutions of* (1).

Proof of the theorem. For any nontrivial solution $x(t)$ of (1), let $v(t) = x^2(t) + a^{-1}(t)[x'(t)]^2$. The theorem will be proved if we show that $v(t) \to 0$ as $t \to +\infty$. Suppose that $v(t) \not\to 0$. Then, since $v'(t) \leq 0$ and $v(t) > 0$, we see that $v(t) \downarrow s > 0$ as $t \uparrow \infty$. Let $\varepsilon > 0$. Then there exists $t_0 = t_0(\varepsilon) \geq 0$ such that

$$s + \varepsilon \geq v(t) > s \quad \text{for all } t \geq t_0. \tag{15}$$

Since $x'' = -ax$,

$$(qv)' \equiv (1-\gamma)q'v - \gamma q'(xx')'a^{-1} + a^{-1}q(x')^2\left(2\gamma\frac{q'}{q} - \frac{a'}{a}\right). \tag{16}$$

If (6) holds for some constant $\gamma > 0$, it holds for all constants γ^* $(0 \leq \gamma^* \leq \gamma)$. Therefore we may assume without loss of generality that $\gamma \leq 1$. By integrating (16) and taking (6) and (15) into consideration, we find that

$$q(t)s - q(t_0)v(t_0) \leq (1-\gamma)(s+\varepsilon)[q(t) - q(t_0)] - \gamma\int_{t_0}^{t} q'a^{-1}(xx')'\,d\tau + o(q(t)) \quad \text{as } t \to \infty. \tag{17}$$

Integrating by parts, we obtain the relation

$$\int_{t_0}^{t} q'a^{-1}(xx')'\,d\tau = q'a^{-1}xx'\Big]_{t_0}^{t} - \int_{t_0}^{t} q''a^{-1}xx'\,d\tau + \int_{t_0}^{t} q'a'a^{-2}xx'\,d\tau. \tag{18}$$

Since $2|a^{-1/2}xx'| \leq x^2 + a^{-1}(x')^2 \leq s + \varepsilon$ and

$$\frac{q'(t)}{a^{1/2}(t)} \leq \int_{t_0}^{t} \frac{|q''(\tau)|}{a^{1/2}(\tau)}\,d\tau + O(1), \qquad \int_{t_0}^{t} \frac{q'(\tau)a'(\tau)}{a^{3/2}(\tau)}\,d\tau \leq 2\int_{t_0}^{t} \frac{|q''(\tau)|}{a^{1/2}(\tau)}\,d\tau + O(1),$$

substitution into (18) shows that

$$\left|\int_{t_0}^{t} q'a^{-1}(xx')'\,d\tau\right| \leq 2(s+\varepsilon)\int_{t_0}^{t} a^{-1/2}|q''|\,d\tau + O(1) \quad \text{as } t \to +\infty. \tag{19}$$

Combining (5), (17), and (19), and letting $t \to \infty$, we obtain the inequality

$$s \leq (1-\nu)(s+\varepsilon) + 2\nu(s+\varepsilon)\lambda,$$

which is possible only if

$$\varepsilon \geq \frac{(1-2\lambda)s\nu}{1-(1-2\lambda)\nu} > 0. \tag{20}$$

Since $\varepsilon > 0$ is arbitrary, (20) contradicts the assumption that $s > 0$, and this proves the theorem.

Proof of the corollaries. Corollary 1 follows from the theorem if we take $q(t) = 1 + \int_0^t p^{-1}(\tau)\,d\tau$. To verify that (5) holds in this case, we substitute $|p'(t)| = p'(t) + 2[-p'(t)]_+$ and integrate by parts to obtain the inequality

$$\int_0^t a^{-1/2}(\tau)\,|q'(\tau)|\,d\tau \leq p^{-1}(0)a^{-1/2}(0) + 2\int_0^t p^{-2}(\tau)a^{-1/2}(\tau)[-p'(\tau)]_+\,d\tau.$$

That the integral on the right is $o(q(t))$ as $t \to \infty$, and hence that (5) holds with $\lambda = 0$, follows from (9) by L'Hospital's rule. The limit in (6) is zero for each choice of ν, since by (8) and (10) there exists a constant T such that

$$2\nu\frac{q'}{q} - \frac{a'}{a} = \frac{q'}{q}\left[2\nu - \frac{a'}{a}p\cdot\left(\int_0^t p^{-1}d\tau + 1\right)\right] \leq 0 \qquad \text{for all } t \geq T.$$

Corollaries 2, 3, and 4 are direct consequences of Corollary 1 with

$$p(t) = a(t), \qquad p(t) = (t+2)\log(t+2), \qquad p(t) = a(t)/a'(t),$$

respectively. Corollary 5 follows from Corollary 3 and the observation that the condition $t a''(t) + a'(t) \geq 0$ implies

$$a(t) - a(t_0) = \int_{t_0}^t \tau a'(\tau)\frac{1}{\tau}\,d\tau \leq t a'(t)\log t - t_0 a'(t_0)\log t_0 \qquad (t \geq \max(t_0, 1));$$

hence, equation (12) holds.

Remark 1. Since integration is a "smoothing" operation, the oscillation of the successive functions $a(t)$, $a'(t)$, and $a''(t)$ is generally progressively more pronounced. Hence, conditions not involving $a''(t)$ directly are probably more practical than those containing $a''(t)$. For instance, Corollaries 4 and 5 do not cover the example

$$a(t) = t^2 + \int_0^t \cos(s^4)\,ds,$$

but Corollary 3 does.

Remark 2. Corollaries 4 and 5 settle the so-called concave and convex cases; that is, either $a'' \leq 0$ or $a'' \geq 0$ implies that $x(t) \to 0$ as $t \to \infty$ for all solutions of (1).

Remark 3. We conjecture that even when

$$\lim_{t\to\infty} a^{-1}(t)a'(t)p(t) = 0,$$

where $p(t)$ is a function satisfying (8) and (9), then $x(t) \to 0$ as $t \to \infty$ for all solutions of (1). In other words, a necessary condition for (1) to have a solution $x(t)$ such that $\limsup_{t\to\infty} |x(t)| > 0$ might be

$$\limsup_{t\to\infty} \frac{a'(t)p(t)}{a(t)} > 0 \quad \textit{and} \quad \liminf_{t\to\infty} \frac{a'(t)p(t)}{a(t)} = 0 .$$

Remark 4. If in the assumptions of Corollary 1, equation (9) is replaced by $p' \geq 0$, then this modified set of conditions is enough to imply that the Sansone condition given by equation (4) holds. To see this, let $\{t_n\}$ be a Sansone sequence, and let

$$\frac{a'(\tau_n)}{a(\tau_n)} = \min_{t_n \leq t \leq t_{n+1}} \frac{a'(t)}{a(t)} \quad (t_n \leq \tau_n \leq t_{n+1}).$$

Then

$$\sum_{n=1}^{\infty} (t_{n+1} - t_n) \frac{a'(\tau_n)}{a(\tau_n)} \geq C \sum_{n=N}^{\infty} \frac{t_{n+1} - t_n}{p(\tau_n)} \geq C \sum_{n=N}^{\infty} \frac{t_{n+2} - t_{n+1}}{p(t_{n+1})} \cdot \frac{t_{n+1} - t_n}{t_{n+2} - t_{n+1}}$$

$$\geq C \int_{t_{N+1}}^{\infty} \frac{dt}{p(t)} = \infty .$$

This offers an alternate way for proving Corollaries 2, 3, and 5.

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