PRIMITIVE INVARIANTS AND CONJUGATE CLASSES OF FUNDAMENTAL REPRESENTATIONS OF A COMPACT SIMPLY CONNECTED LIE GROUP

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Let G be a compact, connected, and simply connected Lie group. The rational cohomology algebra of G can be expressed as an exterior algebra

$$H^*(G; Q) = \Lambda_Q(x_1, \cdots, x_\ell),$$

where $\deg x_i = 2m_i - 1$ $(1 \leq i \leq \ell)$ and $\ell = \text{rank } G$. The integers m_i are called the *primitive invariants* of G.

Among the complex representations of G there are ℓ fundamental ones, $\rho_1, \cdots, \rho_\ell$ that are irreducible, map a fixed maximal torus T of G onto diagonal matrices, and have highest weights $\omega_1, \cdots, \omega_\ell$ satisfying the relations

$$2\langle\omega_i, \alpha_j\rangle / \langle\alpha_j, \alpha_j\rangle = \delta_{ij} \qquad (1 \leq i, j \leq \ell)$$

for a simple system of roots $\{\alpha_1, \cdots, \alpha_\ell\}$ of G with respect to T. The set $\{\rho_1, \cdots, \rho_\ell\}$ is closed under conjugation up to equivalence; that is, the conjugate representation $\bar{\rho}_i$ of ρ_i is equivalent to some ρ_j, so that it can be decomposed into conjugate classes. Each class consists of one self-conjugate representation or two mutually conjugate representations.

In this note we give a general proof of the following theorem.

THEOREM. *The number of even primitive invariants of* G *is equal to the number of conjugate classes of fundamental representations.*

(Professor Armand Borel has pointed out to the author that this theorem can be proved by a method that does not involve KO theory, but still uses Atiyah's result [1].)

The theorem partially illustrates the phenomenon that the number of even primitive invariants is generally larger than the number of odd primitive invariants. (From this point of view, the case $G = SU(2n+1)$ looks rather exceptional.) When no simple factor of G has outer automorphisms, then every ρ_i is self-conjugate, hence every m_i is even.

1. Our basic tool is the rational KO-cohomology, that is, a cohomology theory assigning to each space X (finite CW-complex) the groups $KO^i(X) \otimes Q$ $(i \in Z)$, which we denote by KO_Q [2]. Let

$$\varepsilon: KO^i(X) \to K^i(X) \quad \text{and} \quad \rho: K^i(X) \to KO^i(X) \qquad (i \in Z)$$

denote natural transformations induced by complexification and real restriction of vector bundles respectively [2], [3]. Let g be the generator of K^{-2}(a point) given by the reduced Hopf bundle, let λ and μ be the generators of KO^{-4}(a point) and

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KO^{-8}(a point) such that $\rho(g^2) = \lambda$ and $\varepsilon(\mu) = g^4$ [3]. Then $\lambda^2 = 4\mu$, and the multiplication (tensor product) with μ gives the Bott periodicity isomorphism β of KO.

Put $\lambda' = \lambda/2 \in KO^{-4}(\text{a point}) \otimes Q$. Now we easily see that multiplication with λ' gives a periodicity isomorphism

$$\beta': KO^i(X) \otimes Q \simeq KO^{i-4}(X) \otimes Q \qquad (i \in Z)$$

of period 4, which we may call the Bott periodicity isomorphism of KO_Q. Since $\lambda'^2 = \mu$, β' is compatible with β in the sense that β'^2 is the isomorphism induced by β. Identifying $KO^i(X) \otimes Q$ with $KO^{i-4}(X) \otimes Q$ by β', we obtain a multiplicative Z_4-graded cohomology theory, which we denote by KO'_Q; that is,

$$KO'_Q(X) = \sum_{i=0}^{3} KO^{-i}(X) \otimes Q.$$

Put

$$ch_R = ch \circ \varepsilon: KO \to H^*(\ ; Q).$$

Since ch and ε are additive and multiplicative natural transformations, ch_R also has this property. We can also define ch_R on KO_Q by putting $ch_R = ch_R \otimes 1_Q$. Then

$$ch_R(\lambda') = ch(g^2) = 1 \in H^0(\text{a point}; Z).$$

Hence, for any $x \in KO^i(X) \otimes Q$,

$$ch_R(x) = ch_R(\beta'(x));$$

this means that ch_R induces an additive and multiplicative natural transformation

$$ch_R: KO'_Q \to H^*(\ ; Q),$$

which we also denote by the same symbol. Since KO'_Q(a point) is isomorphic to Q and $ch_R(1) = 1$, we have an isomorphism

$$ch_R: KO'_Q(\text{a point}) \simeq H^*(\text{a point}; Q).$$

Thus, by a general argument using spectral sequences [2], [4, Anhang, pp. A1-A14], we see that

$$ch_R: KO'_Q(X) \simeq H^*(X; Q) \qquad (1)$$

for each finite CW-complex X. Moreover, observing that

$$ch_R(KO^i(X)) \subset \sum_n H^{4n+i}(X; Q)$$

and giving $H^*(X; Q)$ a Z_4-grading by $\sum_n H^{4n+i}(X; Q)$ $(-3 \leq i \leq 0)$, we see that the above isomorphism (1) is degree-preserving.

The relation (1) and the multiplicativity of ch_R imply the Künneth isomorphism

$$KO'_Q(X) \otimes KO'_Q(Y) \simeq KO'_Q(X \times Y)$$

for any finite CW-complexes X and Y.

2. Now we consider the case where X is a compact simply connected Lie group G. By the isomorphism (1), $KO'_Q(G)$ is an exterior algebra over Q of dimension 2^ℓ. Since (1) is degree-preserving, it induces a degree-preserving isomorphism of modules of indecomposable elements (that is, reduced cohomologies modulo the subgroups of decomposable elements), and each one of these groups is degreewise isomorphic to a subgroup generated by a set of generators of the exterior algebra. Hence we see that the following statement holds.

(2) *In any set of generators of the exterior algebra* $KO'_Q(G)$, *the number of generators of degree* -1 *(or* -3*) is equal to the number of even (or odd) primitive invariants of* G.

Let $\alpha\colon G \to U(n)$ be a complex representation of G. The representation α, followed by the inclusion $U(n) \subset U$, gives an element $\iota(\alpha) \in [G, U] = K^{-1}(G)$. Atiyah [1] proved that

$$ch(\iota(\rho_1) \cdots \iota(\rho_\ell))[G] = 1.$$

In particular,

(3) $$ch(\iota(\rho_1) \cdots \iota(\rho_\ell)) \neq 0.$$

By reindexing, we may assume that $\rho_1, \cdots, \rho_s$ are self-conjugate and $\{\rho_{s+i}, \rho_{s+m+i}\}$ $(1 \leq i \leq m$ and $m = (\ell - s)/2)$ form conjugate classes. Then s + m is the number of conjugate classes of fundamental representations. Let

$$y_i = \rho(\iota(\rho_i)) \in KO^{-1}(G) \qquad (1 \leq i \leq s+m),$$

$$y_{s+m+i} = \rho(g \cdot \iota(\rho_{s+i})) \in KO^{-3}(G) \qquad (1 \leq i \leq m).$$

Then

$$ch_R(y_i) = 2 \cdot ch(\iota(\rho_i)) \qquad (1 \leq i \leq s),$$

$$ch_R(y_{s+i}) = ch(\iota(\rho_{s+i}) + \iota(\rho_{s+m+i})) \qquad (1 \leq i \leq m),$$

$$ch_R(y_{s+m+i}) = ch(\iota(\rho_{s+i}) - \iota(\rho_{s+m+i})) \qquad (1 \leq i \leq m).$$

Hence

$$ch_R(y_1 \cdots y_\ell) = \pm 2^{s+m} \cdot ch(\iota(\rho_1) \cdots \iota(\rho_\ell)).$$

Thus, by (3) and (1),

$$y_1 \cdots y_\ell \neq 0$$

in $KO'_Q(G)$, which implies that $y_1, \cdots, y_\ell$ generate an exterior subalgebra of dimension 2^ℓ since the y_i have odd degrees. Finally, by the dimensionality argument, we obtain the relation

(4) $$KO'_Q(G) = \Lambda_Q(y_1, \cdots, y_\ell),$$

where the right-hand member has exactly s + m generators y_i $(1 \leq i \leq s+m)$ of degree -1.

The proposition (2) and the relation (4) prove the theorem.

3. Our theorem implies the following two corollaries.

COROLLARY 1. *The primitive invariants of* G *are all even if and only if every fundamental representation is self-conjugate.*

For a homogeneous space G/K of compact groups, it is well known [5] that the Euler characteristic $\chi(G/K)$ is not zero if and only if rank G = rank K. Hence, for a symmetric pair (G, K) of a compact semisimple group G, $\chi(G/K) \neq 0$ if and only if the involution σ of the pair is an inner automorphism of G, which is equivalent to saying that $\rho_i \circ \sigma \sim \rho_i$ for all fundamental representations ρ_i of the universal covering group $\tilde{G}$ of G. When the dual noncompact group of (G, K) is a Chevalley group, that is, when its Lie algebra is the normal form of the complexification, then $\rho_i \circ \sigma \sim \bar{\rho}_i$ for all ρ_i, since both σ and the conjugation transform every weight to its negative, for a suitable T. Thus we obtain from Corollary 1 a general proof of a remark of Ono [6]. (The author is indebted to Professor Hans Samelson, who pointed out Ono's remark to him.)

COROLLARY 2. *Let* (G, K) *be a symmetric pair of a compact, semisimple Lie group* G *such that its dual noncompact group is a Chevalley group. Then* $\chi(G/K) \neq 0$ *if and only if the primitive invariants of* G *are all even.*

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