INTEGRABILITY CONDITIONS FOR ALMOST-COMPLEX MANIFOLDS

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The integrability condition $\bar{\partial}^2 = 0$ characterizes those almost complex structures that arise from complex-analytic structures, and it should in some way be reflected in corresponding conditions on Riemannian metrics; that is, we should be able to distinguish between metrics that are almost-hermitian with respect to integrable almost-complex structures, and those that are almost-hermitian with respect to non-integrable structures. This paper introduces a real-valued function (on the tangent vectors of the manifold in question) whose positivity (or lack of it) permits us to make the distinction.

More precisely, let J be an almost-complex structure on the 2n-dimensional manifold M, and let g(X, Y) be an almost-hermitian metric on M such that

$$g(JX, JY) = g(X, Y).$$

We denote by B the bundle of almost-complex frames of M, by ω the restriction to B of the Riemannian connection of M, and by $\mathfrak{M}$ the orthogonal complement to the Lie algebra of U(n) in the Lie algebra of O(2n). Let Δ denote the $\mathfrak{M}$-component of ω, and finally, for each tangent vector X of M, let

$$\sigma_g(X) = -\operatorname{trace} \operatorname{Im} [\Delta(X), \Delta(JX)]$$

(see Section 3). We shall prove in Section 5 that σ_g is nonnegative whenever J is integrable, and in Section 6 that σ_g vanishes identically if and only if g is a Kähler metric.

1. THE METRICS

Let M be an almost-complex manifold of real dimension 2n, with an almost-complex operator J: $J^2 = -I$. Let g(X, Y) be a Riemannian metric on M that is compatible with J. That is, let

$$g(JX, JY) = g(X, Y)$$

for each pair of tangent vectors X and Y of M; or equivalently, let

$$g(JX, Y) = -g(X, JY)$$

for each pair of tangent vectors X and Y of M. Such a metric g is called *almost-hermitian*, and its *fundamental form* is the real-valued 2-form

$$\Omega(X, Y) = -(4\pi)^{-1} \cdot g(X, JY).$$

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The almost-complex structure J of M is called *integrable* if it arises from a complex-analytic structure of M. The metric g is called a *Kähler metric* if J is integrable and if the fundamental form of g is closed:

$$d\Omega = 0.$$

The *bundle of frames* F(M) of M is the set of all (2n + 1)-tuples $(m, e_1, \cdots, e_{2n})$, where m is a point of M and $e_1, \cdots, e_{2n}$ are tangent vectors at m satisfying the conditions $g(e_i, e_j) = \delta_{ij}$. This bundle is reducible to a principal U(n)-bundle over M, which we denote by B; in other words, B consists of all points $(m, e_1, \cdots, e_{2n})$ of F(M) that have the additional properties

$$Je_i = e_{n+i} \quad \text{and} \quad Je_{n+i} = -e_i \quad (1 \le i \le n).$$

λ will denote the natural projection of F(M) onto M.

2. THE CONNECTIONS

Let o(2n) denote the Lie algebra of O(2n), and let u(n) denote the Lie algebra of U(n). The set o(2n) then consists of all real skewsymmetric $2n \times 2n$ matrices, and u(n) of all real $2n \times 2n$ matrices of the form

$$\left[\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right],$$

where A and B are $n \times n$ matrices satisfying the conditions $A^t = -A$ and $B^t = B$. Let $\mathfrak{M}$ be the set of all real $2n \times 2n$ matrices of the form

$$\left[\begin{array}{c|c} A & B \\ \hline B & -A \end{array}\right]$$

where A and B are $n \times n$ matrices satisfying the conditions $A^t = -A$ and $B^t = -B$. Then $\mathfrak{M}$ is orthogonal to u(n) with respect to the Killing metric of O(2n), and o(2n) is the sum of $\mathfrak{M}$ and u(n):

$$o(2n) = u(n) + \mathfrak{M}, \qquad \operatorname{ad} U(n)(\mathfrak{M}) \subset \mathfrak{M}.$$

Let w be the Riemannian connection of M. Then w is an o(2n)-valued 1-form on F(M). We denote by p the projection of o(2n) onto u(n) with respect to the decomposition $o(2n) = u(n) + \mathfrak{M}$. Let ω denote the restriction of w to the bundle B, let $\omega_o = p \circ \omega$, and let $\Delta = \omega - \omega_o$. It is known that *the 1-form* ω_o *is a connection on the bundle* B, and that *the metric* g *is a Kähler metric if and only if* Δ *vanishes identically on* B.

Certain vector fields $E^1, \cdots, E^{2n}$ on B, associated with the connection ω_o, are defined in the following way. If $\mathfrak{m} = (m, e_1, \cdots, e_{2n})$ is a point of B, then $E^i(\mathfrak{m})$ is the unique tangent vector at $\mathfrak{m}$ satisfying the conditions

$$\lambda_*(E^i(\mathfrak{m})) = e_i, \qquad \omega_o(E^i(\mathfrak{m})) = 0.$$

These make it possible to define a linear operator J on M, in the following way: If t is a tangent vector of B with $\lambda_*(t) = 0$, then $J(t) = 0$; moreover, $J(E^i) = E^{n+i}$ and $J(E^{n+i}) = -E^i$. Hence $\lambda_* \circ J = J \circ \lambda_*$.

3. THE FORM σ_g

The matrices of u(n) have two interpretations. They can be considered either as complex $n \times n$ matrices of the form $A + \sqrt{-1} \cdot B$, where A and B are real $n \times n$ matrices satisfying the conditions $A^t = -A$ and $B^t = B$, or else as real $2n \times 2n$ matrices of the kind described in Section 2. The second interpretation is the one that makes u(n) a subgroup of o(2n); the first makes the following definition reasonable: If $\theta = (\theta_{ij})$ is an element of u(n), then

$$\text{trace Im } \theta = \sum_{k=1}^{n} \theta_{k,n+k}.$$

We extend this formally to o(2n), that is, if $\theta = (\theta_{ij})$ is an element of o(2n), we define trace Im $\theta = \sum_{k=1}^{n} \theta_{k,n+k}$. Thus trace Im is a linear real-valued function on o(2n), invariant under the adjoint action of U(n). It should be observed that

$$\text{trace Im } \theta = 0 \quad \text{if } \theta \text{ is in } \mathfrak{M}.$$

Finally, we define an o(2n)-valued 2-form $[\triangle, \triangle]$ on B: If t and t' are tangent vectors of B, let

$$[\triangle, \triangle](t, t') = \frac{1}{2} \cdot (\triangle(t) \cdot \triangle(t') - \triangle(t') \cdot \triangle(t)).$$

Thus the real-valued 2-form trace Im $[\triangle, \triangle]$ on B is invariant under the right-action of U(n), and it is horizontal (that is, it vanishes on any pair of vectors one of which is in the nullspace of λ_*). This means that it can be dropped to a real-valued 2-form on M, which will also be denoted by trace Im $[\triangle, \triangle]$. We let

$$\sigma_g(X) = -(\text{trace Im } [\triangle, \triangle])(X, JX)$$

for each tangent vector X of M. Then σ_g is real-valued and homogeneous of degree 2.

4. THE INTEGRABILITY CONDITIONS

Real-valued 1-forms $\phi_1, \cdots, \phi_{2n}$ can be defined on B in the following way: If t is a tangent vector at a point $(m, e_1, \cdots, e_{2n})$ of B, then $\lambda_*(t)$ is a tangent vector at the point m of M and hence is a linear combination $\sum_{i=1}^{2n} c_i \cdot e_i$ of the vectors $e_1, \cdots, e_{2n}$; we let $\phi_i(t) = c_i$ for $1 \leq i \leq 2n$. As we mentioned previously, the almost-complex structure J of M is said to be integrable if it arises from a complex-analytic structure of M. In terms of the bundle B, this means that the points $(m, e_1, \cdots, e_{2n})$ of B have the following property:

The vectors $e_i - \sqrt{-1} \cdot e_{n+i}$ of M $(1 \leq i \leq n)$ are holomorphic vectors. We can define complex-valued 1-forms $\hat{\phi}_1, \cdots, \hat{\phi}_{2n}$ on B by setting

$$\hat{\phi}_i = \phi_i + \sqrt{-1}\cdot\phi_{n+i} \quad \text{and} \quad \hat{\phi}_{n+i} = \phi_i - \sqrt{-1}\cdot\phi_{n+i} \quad (1 \le i \le n).$$

The forms $\hat{\phi}_i$ (respectively, $\hat{\phi}_{n+i}$) are called *homogeneous of type* (1, 0) (respectively, *homogeneous of type* (0, 1)). More generally, a form

$$\hat{\phi}_{i_1} \wedge \cdots \wedge \hat{\phi}_{i_r} \wedge \hat{\phi}_{j_1} \wedge \cdots \wedge \hat{\phi}_{j_s}$$

on B is called *homogeneous of type* (r, s) if $1 \le i_1, \cdots, i_r \le n < j_1, \cdots, j_s \le 2n$, and an (r + s)-form on B is called a *form of type* (r, s) if it is a linear combination (over the ring of functions on B) of homogeneous forms of type (r, s).

An alternate interpretation of type is often useful. A vector $t - \sqrt{-1}\cdot Jt$ of B is called of *type* (1, 0) if t is a real tangent vector of B with $\omega_o(t) = 0$. Similarly, a vector $t + \sqrt{-1}\cdot Jt$ of B with $\omega_o(t) = 0$ is called of *type* (0, 1). An (r + s)-form β on B is of *type* (r, s) if and only if the following conditions are satisfied:

1. β is horizontal; that is, it vanishes whenever one of its vector arguments lies in the nullspace of λ_*.

2. If each of the vectors $\hat{t}_1, \cdots, \hat{t}_{r+s}$ is either of type (1, 0) or of type (0, 1), then $\beta(\hat{t}_1, \cdots, \hat{t}_{r+s}) = 0$ unless exactly r of the vectors $\hat{t}_k$ are of type (1, 0), and exactly s are of type (0, 1).

A tangent vector t of B is called *vertical* if $\lambda_*(t) = 0$, and it is called *horizontal* if $\omega_o(t) = 0$. Therefore every tangent vector t of B is the sum of a unique vertical vector $V_o(t)$ and a unique horizontal vector $H_o(t)$. The *covariant differential* $D_o\beta$ of a k-form β on B is defined by the rule that, for each set $t_1, \cdots, t_{k+1}$ of tangent vectors of B,

$$D_o\beta(t_1, \cdots, t_{k+1}) = d\beta(H_o(t_1), \cdots, H_o(t_{k+1})).$$

In particular, it is known that if k = r + s, β is of type (r, s), and J is integrable, then $D_o\beta$ is the sum of a form of type (r + 1, s) and a form of type (r, s + 1). Henceforth, *we assume that* J *is integrable.*

5. SEVERAL LEMMAS

LEMMA 1. *The forms* $D_o\hat{\phi}_i$ *are of type* (1, 1) $(1 \le i \le n)$.

Proof. The Riemannian connection ω on B is torsion-free; that is,

$$d\phi_i = \sum_{r=1}^{2n} \omega_{ir} \wedge \phi_r \quad (1 \le i \le 2n).$$

Thus, for $1 \le i \le n$,

$$d\hat{\phi}_i = d\phi_i + \sqrt{-1}\cdot d\phi_{n+i} = \sum_{r=1}^{2n} \omega_{ir} \wedge \phi_r + \sqrt{-1}\cdot \sum_{r=1}^{2n} \omega_{n+i,r} \wedge \phi_r$$

$$= \sum_{j=1}^{n} (\omega_{ij} \wedge \phi_j + \omega_{i,n+j} \wedge \phi_{n+j} + \sqrt{-1}\cdot\omega_{n+i,j} \wedge \phi_j + \sqrt{-1}\cdot\omega_{n+i,n+j} \wedge \phi_{n+j}).$$

It follows that for $1 \leq i \leq n$,

$$D_o \hat{\phi}_i = \sum_{j=1}^{n} (\Delta_{ij} \wedge \phi_j + \Delta_{i,n+j} \wedge \phi_{n+j} + \sqrt{-1} \cdot \Delta_{n+i,j} \wedge \phi_j + \sqrt{-1} \cdot \Delta_{n+i,n+j} \wedge \phi_{n+j}).$$

However, the form $\Delta = (\Delta_{ij})$ is $\mathfrak{M}$-valued, and therefore

$$\Delta_{n+i,n+j} = -\Delta_{ij}, \qquad \Delta_{i,n+j} = \Delta_{n+i,j}.$$

Consequently,

$$D_o \hat{\phi}_i = \sum_{j=1}^{n} (\Delta_{ij} \wedge \phi_j + \Delta_{i,n+j} \wedge \phi_{n+j} + \sqrt{-1} \cdot \Delta_{i,n+j} \wedge \phi_j - \sqrt{-1} \cdot \Delta_{ij} \wedge \phi_{n+j})$$

$$= \sum_{j=1}^{n} (\Delta_{ij} + \sqrt{-1} \cdot \Delta_{i,n+j}) \wedge \hat{\phi}_{n+j}.$$

To compare types, we note first that the 1-form $\hat{\phi}_i$ is of type (1, 0), so that the 2-form $D_o \hat{\phi}_i$ is the sum of a form of type (1, 1) and a form of type (2, 0). On the other hand,

$$D_o \hat{\phi}_i = \sum_{j=1}^{n} (\Delta_{ij} + \sqrt{-1} \cdot \Delta_{i,n+j}) \wedge \hat{\phi}_{n+j};$$

therefore $D_o \hat{\phi}_i$ is the sum of a form of type (1, 1) and a form of type (0, 2), because $\hat{\phi}_{n+j}$ is of type (0, 1). It follows that $D_o \hat{\phi}_i$ must be of type (1, 1), as required.

We notice as a first consequence that the form $(\Delta_{ij} + \sqrt{-1} \cdot \Delta_{i,n+j})$ must be of type (1, 0). Applying this to the vectors of B of type (0, 1), we obtain the following lemma.

LEMMA 2. *Let* I' *be the real* $2n \times 2n$ *matrix* $(\delta_{j,n+i} - \delta_{i,n+j})$. *Then* $\Delta(Jt) = -I' \cdot \Delta(t)$ *for each tangent vector* t *of* B.

Proof. It suffices to consider vectors t with $\omega_o(t) = 0$, because the form Δ vanishes on the nullspace of λ_*. Let t be such a vector. Then $t + \sqrt{-1} \cdot Jt$ is of type (0, 1), and hence

$$(\Delta_{ij} + \sqrt{-1} \cdot \Delta_{i,n+j})(t + \sqrt{-1} \cdot Jt) = 0.$$

Comparison of real and imaginary parts yields the identities

$$\Delta_{i,n+j}(Jt) = \Delta_{ij}(t), \qquad \Delta_{ij}(Jt) = -\Delta_{i,n+j}(t).$$

Thus

$$\Delta(t) = \left[\begin{array}{c|c} A & B \\ \hline B & -A \end{array}\right], \qquad \Delta(Jt) = \left[\begin{array}{c|c} -B & A \\ \hline A & B \end{array}\right], \qquad I' = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right],$$

and therefore $I' \cdot \Delta(t) = -\Delta(Jt)$.

LEMMA 3. *For each tangent vector* t *of* B, $I' \cdot \Delta(t) = -\Delta(t) \cdot I'$.

Proof. $\Delta(t)$ is in $\mathfrak{M}$, and hence it has the form

$$\left[\begin{array}{c|c} A & B \\ \hline B & -A \end{array}\right],$$

where A and B are skew-symmetric matrices: $A^t = -A$ and $B^t = -B$. Therefore the lemma is an immediate consequence of ordinary matrix multiplication.

LEMMA 4. *If* θ *is an element of* u(n), *then*

$$\text{trace Im } \theta = -\frac{1}{2} \cdot \text{trace } I' \cdot \theta .$$

Proof. θ is in u(n) and hence is a $2n \times 2n$ matrix of the form

$$\left[\begin{array}{c|c} a & b \\ \hline -b & a \end{array}\right],$$

where $a^t = -a$. Therefore

$$\text{trace } I' \cdot \theta = -2 \cdot \sum_{k=1}^{n} b_{k,n+k} = -2 \cdot \text{trace Im } \theta .$$

LEMMA 5. *If* t *is a tangent vector of* B, *then*

$$(\text{trace Im } [\Delta, \Delta])(t, Jt) = \text{trace } \Delta(t) \cdot \Delta(t).$$

Proof. Since the bracket of any two matrices of $\mathfrak{M}$ lies in u(n), we can invoke Lemma 4 for the matrix $\theta = [\Delta(t), \Delta(Jt)]$ of u(n). Thus

$$\begin{aligned} (\text{trace Im } [\Delta, \Delta])(t, Jt) &= \text{trace Im } [\Delta(t), \Delta(Jt)] \\ &= -\frac{1}{2} \cdot \text{trace } I' \cdot [\Delta(t), \Delta(Jt)] \\ &= \frac{1}{2} \cdot \text{trace } I' \cdot [\Delta(t), I' \cdot \Delta(t)] \qquad \text{(Lemma 2)} \\ &= \frac{1}{2} \cdot \text{trace } (I' \cdot \Delta(t) \cdot I' \cdot \Delta(t) - I' \cdot I' \cdot \Delta(t) \cdot \Delta(t)) \\ &= -\text{trace } (I' \cdot I' \cdot \Delta(t) \cdot \Delta(t)) = \text{trace } \Delta(t) \cdot \Delta(t). \end{aligned}$$

LEMMA 6. *If* t *is a tangent vector of* B, *then* trace $\Delta(t) \cdot \Delta(t) \leq 0$.

Proof. Since trace $\Delta(t) \cdot \Delta(t) = 2 \cdot \text{trace } (A^2 + B^2)$ and $A^t = -A$ and $B^t = -B$, we obtain the relations

$$\text{trace } \Delta(t) \cdot \Delta(t) = \sum_{i,j=1}^{n} (a_{ij} a_{ji} + b_{ij} b_{ji}) = \sum_{i,j=1}^{n} (-a_{ij}^2 - b_{ij}^2) \leq 0.$$

Here $A = (a_{ij})$ and $B = (b_{ij})$.

COROLLARY. *If* J *is integrable and* t *is a tangent vector of* B, *then*

$$(\text{trace Im } [\Delta, \Delta])(t, Jt) \leq 0.$$

Proof. This is an immediate consequence of Lemmas 5 and 6.

6. KÄHLER METRICS

PROPOSITION. *If* J *is integrable, then* g *is a Kähler metric if and only if* σ_g *vanishes on* M.

Proof. If g is a Kähler metric, then $\Delta = 0$ and therefore $\sigma_g = 0$ on M. Suppose on the other hand that J is integrable and that $\sigma_g = 0$ on M. Let t be a tangent vector of B, and let $X = \lambda_*(t)$. Then

$$0 = \sigma_g(X) = -(\text{trace Im } [\Delta, \Delta])(X, JX) = -(\text{trace Im } [\Delta, \Delta])(t, Jt)$$

$$= -\text{trace } \Delta(t) \cdot \Delta(t) \qquad \text{(Lemma 5)}$$

$$= \sum_{i,j=1}^{n} (a_{ij}^2 + b_{ij}^2) \qquad \text{(Lemma 6)},$$

and it follows that $a_{ij} = b_{ij} = 0$ for $1 \leq i, j \leq n$. Thus $\Delta(t) = 0$. Therefore Δ vanishes identically on B, and consequently g is a Kähler metric.

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