A NOTE ON INSEPARABILITY

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1. INTRODUCTION

It is well known that for certain fields E there exist finite extensions P/E that are not separable, yet contain no purely inseparable elements. However, as far as the author has been able to determine, the question of when such a phenomenon occurs has not been discussed in the literature; various of the well-known books give only examples (see [1, p. 136, Exercise 17] and [2, p. 49, Exercise 3]; the example in the first printing of [2] seems mistaken, but the author understands that this will be corrected). It seems natural to ask for more general information, and it is the object of this note to make some remarks in this direction. Since this paper was written, it has come to the author's attention that R. W. Gilmer, Jr., W. J. Heinzer, and H. F. Kreimer are preparing a joint paper in which similar problems are considered.

We write P/E to indicate that P is a finite-dimensional extension field of E. The letter p is reserved for the characteristic of the fields under discussion, and since our questions generally evaporate when $p = 0$, we shall always assume that $p > 0$. We write $\Sigma(P/E)$ to denote the subfield of P consisting of all elements separable over E, and we write Σ/E to denote a separable extension of E. For lack of a better name, we call an extension P/E *exceptional* if it is not separable yet contains no purely inseparable elements. We say that a separable extension Σ/E is a *realizable* extension of E provided there exists an exceptional extension P/E such that $\Sigma = \Sigma(P/E)$. Otherwise, our terminology is that of [2].

2. A CRITERION

THEOREM 2.1. *A separable extension* Σ/E *is a realizable extension if and only if*

$$\bigcup_{x \in E} \Sigma^p(x) \neq \Sigma .$$

Proof. Suppose first that $\Sigma = \Sigma(P/E)$ for some exceptional extension P/E. Then P/Σ is purely inseparable, so that for each $y \in P$ there exists an m such that $y^{p^m} \in \Sigma$. We may thus choose $z \in P$, $z \notin \Sigma$, with $z^p \in \Sigma$. Should $z^p \in \Sigma^p(x)$ for some $x \in E$, then, since the mapping $y \to y^p$ is one-to-one on P and $z \in \Sigma$, so that $z^p \notin \Sigma^p$, we see that $\Sigma^p(x) \neq \Sigma^p$. Hence both x and z^p have degree p over Σ^p; thus $\Sigma^p(x) = \Sigma^p(z^p)$, and

$$x \in \Sigma^p(z^p) \cap E \leq P^p \cap E .$$

Finally, since P/E is exceptional, it follows that $P^p \cap E = E^p$, so that $x \in E^p \leq \Sigma^p$, contrary to the relation $\Sigma^p(x) \neq \Sigma^p$. We thus conclude that $z^p \notin \Sigma^p(x)$ for all $x \in E$; therefore the condition is necessary.

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For the converse, suppose Σ/E is separable and there exists $s \in \Sigma$ such that $s \notin \Sigma^p(x)$ for all $x \in E$. Then, with $P = \Sigma(s^{1/p})$, it is clear that P/Σ is purely inseparable, so that $\Sigma = \Sigma(P/E)$. Suppose that $y^p = x \in E$ for some $y \in P$, and put $z = s^{1/p}$. Then $y = s_0 + s_1 z + \cdots + s_{p-1} z^{p-1}$ for some $s_j \in \Sigma$, which implies

$$x = y^p = s_0^p + s_1^p s + \cdots + s_{p-1}^p s^{p-1} \in \Sigma^p(s)\,.$$

Now if $x \notin \Sigma^p$, then as above, $\Sigma^p(x) = \Sigma^p(s)$, contrary to the choice of s. On the other hand, $x \in \Sigma^p$ implies $y \in \Sigma$, since $y^p = x$. Separability of Σ/E now implies that $y \in E$, as required. Thus P/E is exceptional.

Remark. It is well known that separability of Σ/E is equivalent to the condition $\Sigma^p(E) = \Sigma$. Theorem 2.1 may be interpreted as saying that nonrealizability of the separable extension Σ/E is equivalent to "strong separability" in the sense that

$$\bigcup_{x \in E} \Sigma^p(x) = \Sigma\,.$$

COROLLARY 2.2. *Let Σ/E be separable, and suppose $\Sigma = \Sigma^p(s)$ for some $s \in \Sigma$. Then Σ/E is not realizable.*

Proof. Clearly $\Sigma = \Sigma^p(s)$ implies that either $[\Sigma\colon \Sigma^p] = p$ or $\Sigma = \Sigma^p$. In either case there are no intermediate fields; therefore $\bigcup_{x \in F} \Sigma^p(x) = \Sigma$ for any subfield F of Σ, unless $F \leq \Sigma^p$. Since Σ/E is separable, if $E \leq \Sigma^p$, then $\Sigma = \Sigma^p(E) = \Sigma^p$; therefore Σ is perfect, hence has no inseparable extensions. Thus, either by the theorem or the alternative that Σ is perfect, Σ/E cannot be the separable part of an exceptional extension P/E.

COROLLARY 2.3. *Let E be a field such that $E = E^p(x)$ for some $x \in E$. Then there is no exceptional extension of E.*

Proof. If $E = E^p$, the statement clearly follows. Otherwise, it is obvious that $\Sigma = \Sigma^p(x)$ whenever Σ/E is separable, and we may apply Corollary 2.2.

THEOREM 2.4. *Let K be a field of characteristic p, and x an indeterminate. Then K(x) is realizable (in other words, is the separable part of an exceptional extension P/E) if and only if K is not perfect.*

Proof. If K is perfect, then $K(x)^p = K(x^p)$, so that $K(x) = [K(x)^p](x)$, and by Corollary 2.2, K(x) is not realizable. Suppose now that K is not perfect, and choose $a \in K$ so that $a \notin K^p$. Put

$$P = K(x), \qquad E = K(y), \qquad \text{where } y = x^{p^2}(x^p + a)^{-1}\,.$$

It suffices to show that P/E is exceptional, since then $\Sigma(P/E) = K(x^p)$ and is therefore isomorphic to K(x).

By [2, Theorem 7, p. 158] we see that $[P\colon E] = p^2$, and $\lambda^{p^2} - y\lambda^p - ya$ is the irreducible polynomial for x over E. Now suppose P/E is not exceptional. Then there exists $b \in P$ such that $b \notin E$ and $b^p \in E$, and $[E(b)\colon E] = p$, hence $E(b) \neq P$ and $[P\colon E(b)] = p$. By Lüroth's Theorem, we may write $E(b) = K(t)$ for some t, and if

$$t = r(x)s(x)^{-1} \qquad (r(x),\ s(x) \in K[x])\,,$$

then $p = \deg t = \max(\deg r(x), \deg s(x))$. Without loss of generality, we may assume that $\deg r(x) = p$ and write

$$r(x) = \sum_{i=0}^{p} a_i x^i \quad (a_p \neq 0), \qquad s(x) = \sum_{i=0}^{p} b_i x^i .$$

Now t^p has degree p^2 and belongs to E, since $K(t)^p = E(b)^p \leq E$. Hence, by [2, Theorem 7, p. 158] again, $[P: K(t^p)] = p^2$; this, with $K(t^p) \leq E$, implies that $E = K(t^p)$. As above, this gives the irreducible polynomial $r(\lambda)^p - t^p s(\lambda)^p$ for x over E. Since t is an indeterminate over K and $a_p \neq 0$, it is clear that $a_p^p - t^p b_p^p \neq 0$. Thus we may divide $r(\lambda)^p - t^p s(\lambda)^p$ by this element of E to obtain the irreducible monic polynomial for x over E; that is,

$$(a_p^p - t^p b_p^p)^{-1}[r(\lambda)^p - t^p s(\lambda)^p] = \lambda^{p^2} - y\lambda^p - ya .$$

Equating coefficients of λ^p, we obtain the equation

$$(a_p^p - t^p b_p^p)^{-1}(a_1^p - t^p b_1^p) = -y = -x^{p^2}(x^p + a)^{-1} .$$

Solving for a, we deduce that $a \in P^p \cap K = K^p$, a contradiction. Thus, P/E contains no purely inseparable elements, and since x is clearly inseparable over E, P/E is exceptional.

COROLLARY 2.5. *If* K *is not perfect, then* K(x) (x *an indeterminate*) *admits exceptional extensions.*

COROLLARY 2.6. *Let* K *be a perfect field, and let* E/K(x) *be separable. Then* E *has no exceptional extensions.*

Proof. It is clear that if P/E were exceptional, then P/K(x) would also be exceptional.

3. SOME FURTHER REMARKS

Let P/E be an arbitrary finite-dimensional extension. Put $\Sigma = \Sigma(P/E)$, and denote by L the subfield of P consisting of all elements purely inseparable over E. If P/L is separable, then [2, p. 50] $P = \Sigma \otimes_E L$. On the other hand, it is clear that if P/L is not separable, then it is an exceptional extension, and in this case $\Sigma(P/L) = \Sigma \otimes_E L$. Thus our observation amounts to saying that arbitrary extensions P/E that do not split into the tensor product of separable part and purely inseparable part always give rise to exceptional extensions P/L, with $\Sigma(P/L) = \Sigma(P/E) \otimes_E L$. The following proposition combines this remark with a converse to it.

PROPOSITION 3.1. *Let* P/E *be an arbitrary finite-dimensional extension, and let* L *be the subfield of purely inseparable elements. Then either* $P = \Sigma(P/E) \otimes_E L$, *or* P/L *is exceptional, with* $\Sigma(P/L) = \Sigma(P/E) \otimes_E L$.

On the other hand, let P/L *be an exceptional extension of the field* L. *Then, for any* $E \leq L$ *such that* $[L: E] < \infty$ *and* L/E *is purely inseparable,*

$$\Sigma(P/L) = \Sigma(P/E) \otimes_E L$$

and L *is the subfield of* P *consisting of all elements purely inseparable over* E.

Proof. Only the second statement remains to be verified. If P/L is exceptional and L/E is purely inseparable, it is clear that L is the set of elements of P purely inseparable over E. Moreover, the hypothesis that $\Sigma(P/L)$ is separable over L and L/E is purely inseparable implies that

$$\Sigma(P/L) = \Sigma(\Sigma(P/L)/E) \otimes_E L$$

(see [2, p. 50]). Finally, $\Sigma(P/E) \leq \Sigma(\Sigma(P/L)/E)$, since $\Sigma(P/E) \leq \Sigma(P/L)$. On the other hand, any element of $\Sigma(P/L)$ that is separable over E clearly lies in $\Sigma(P/E)$, so that $\Sigma(P/E) = \Sigma(\Sigma(P/L)/K)$. Hence $\Sigma(P/L) = \Sigma(\Sigma(P/L)/E \otimes_E L = \Sigma(P/E) \otimes_E L$, as required.

The proposition above may be interpreted to say that exceptional extensions are perhaps not very rare, since if P/E is not normal, it is to be expected that the splitting $P = \Sigma \otimes_E L$ of the first part of Proposition 3.1 does not occur. Normal extensions P/E of course do split [2, p. 52]. One might ask, however, to what extent an extension P/E is representable as a tensor product of its subfield L of purely inseparable elements and another subfield, perhaps larger than $\Sigma(P/E)$. Our next result discusses this question in a special setting. It is convenient to define the *exponent* of an extension P/E (of characteristic $p \neq 0$) to be the exponent of P over $\Sigma(P/E)$. We recall that a field P is said to be a *composite* of two of its subfields F and K provided P is generated by F and K. If P/E is finite-dimensional, and F and K are subfields of P containing E, then (see [2, p. 84]) the field generated by F and K is

$$(F, K) = \left\{ \sum_{i=1}^{m} f_i k_i \,\middle|\, f_i \in F,\ k_i \in K \right\}.$$

THEOREM 3.2. *Let* P/E *be an extension of exponent* 1, *let* $\Sigma = \Sigma(P/E)$, *and let* L *be the subfield of* P *consisting of elements purely inseparable over* E. *If* M *is any subfield of* P *maximal with respect to* $\Sigma \leq M$ *and* $M \cap L = E$, *then* $P = (M, L)$.

Proof. If $M = P$, the theorem is trivial; therefore we assume that $M \neq P$, or equivalently, that $L \neq E$. Then, for each $x \in P$ with $x \notin M$, we can assert that $x^p \in M$, and by maximality of M, there exist $m_i \in M$ and $a \in L$ ($a \notin E$) such that

$$(*) \qquad \sum_{i=0}^{p-1} m_i x^i = a,$$

since $M(x) \cap L \neq E$ and $[M(x): M] = p$. Hence x satisfies the equation

$$\sum_{i=0}^{p-1} m_i \lambda^i - a = 0$$

over (M, L). If $x \notin (M, L)$, then its minimal polynomial over (M, L) is $\lambda^p - x^p$, and the fact that (*) is of degree less than p implies $m_i = 0$ for $i > 0$. Now $m_0 = a \in M \cap L = E$, contrary to the relation $a \notin E$. Thus $x \in (M, L)$; hence, $P = (M, L)$, as required.

We note that if $\Sigma(P/E)$ is already maximal in the sense of Theorem 3.2, then $P = \Sigma \otimes_E L$, since [2, pp. 84-85] any composite of Σ and L is a homomorphic image of $\Sigma \otimes_E L$, and since the hypothesis that Σ/E is separable and L/E is purely inseparable implies that $\Sigma \otimes_E L$ is a field [2, p. 52]. We also remark that if $M \neq \Sigma$, then M/E is exceptional, and in this case $M \otimes_E L$ has a unique maximal ideal J, the set of nonunits, with $(M, L) \cong (M \otimes_E L)/J$ [2, p. 197].

COROLLARY 3.3. *E admits no exceptional extension if and only if every extension P of exponent 1 splits:* $P = \Sigma(P/E) \otimes_E L$.

Proof. If P/E is of exponent 1 and does not split, then by the remarks above we can choose the subfield M in Theorem 3.2 distinct from Σ. Then M/E is exceptional. Conversely, if every extension P/E of exponent 1 splits and K/E is exceptional, then $K/\Sigma(K/E)$ is purely inseparable, hence contains subfields of exponent 1 over $\Sigma(K/E)$. Such subfields are exceptional over E, hence cannot split, contrary to the hypothesis.

COROLLARY 3.4. *Let K be perfect, and let* $E = K(x)$ *(x an indeterminate). Then every extension of E of exponent 1 splits.*

Proof. Corollaries 2.6 and 3.3.

Our final remark concerns the lattice of subfields of an exceptional extension P/E. We call an intermediate field L/E of P/E *distinguished* provided L/E is exceptional and P/L is either separable or exceptional. These conditions are equivalent to

$$L \not\leq \Sigma(P/E) \quad \text{and} \quad P^p \cap L = L^p.$$

Put $\Sigma = \Sigma(P/E)$, and let $\{\Sigma_i \mid i = 1, \cdots, s\}$ be the collection of subfields of Σ that contain E and have the property that P/Σ_i is not exceptional. For $i = 1, \cdots, s$, put

$$L(\Sigma_i) = \{x \in P \mid x^{p^m} \in \Sigma_i \text{ for some } m\}.$$

For each distinguished subfield L/E of P/E, we have the separable part $\Sigma(L/E) = L \cap \Sigma$. The following is now easily established.

PROPOSITION 3.5. *Let P/E be exceptional, let* $\Sigma = \Sigma(P/E)$, *and let* $\{\Sigma_i \mid i = 1, \cdots, s\}$ *be the collection of subfields of* Σ/E *over which P is not exceptional. Then* $\Sigma_i \to L(\Sigma_i)$ *and* $L \to L \cap \Sigma$ *are inverse, inclusion-preserving mappings between the set* $\{\Sigma_i \mid i = 1, \cdots, s\}$ *and the set of distinguished subfields of P/E. In particular, if E is maximal in* $\Sigma(P/E)$ *with respect to the requirement that P/E be exceptional, then there is a one-to-one inclusion-preserving correspondence between distinguished subfields of P/E and subfields of* Σ/E.

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