ON THE ZEROS OF POWER SERIES

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1. An upper bound for the number of zeros.
2. The exact number of zeros.
3. The function $f_\kappa(z) = \sum (n+1)^\kappa z^n$ $(\kappa > 0)$.
4. The function $g_\kappa(z) = \sum (1 - c^{n+1})^\kappa z^n$ $(0 < c < 1,\ \kappa > 0)$.
5. Hurwitz's theorem on the zeros of Bessel functions.

The results of this paper developed from an investigation of the zeros of the function $f_\kappa(z) = \sum_0^\infty (n+1)^\kappa z^n$, which has an analytic continuation into the complex plane with a cut along the real axis from 1 to ∞. The location of the zeros of this function is of importance for questions in Riesz-summability (see for example Riesz [24], Peyerimhoff [21], Kuttner [11], and Miesner [18]). In 1962, Kuttner [11] showed that if $0 \le \kappa < 2$, then $f_\kappa(z) \ne 0$ for $|z| \le 1$; a short proof of this result (Peyerimhoff [22]) used the fact that $f_\kappa(z)$ can be written in the form $f_\kappa(z) = \psi(z) F(z)$, where $\psi(z)$ has no zeros and

$$F(z) = P_{k-1}(z) + \sum_{n=k}^{\infty} z^n p_{n-k} = P_{k-1}(z) + z^k \int_0^1 \frac{dg(t)}{1 - zt}$$

(here $P_k(z)$ is a real polynomial of degree at most k; $\kappa = k + \theta$, $k = 0, 1, \cdots$, $0 < \theta \le 1$; the sequence $\{p_n\}$ is totally monotone, and $g(t)$ is increasing); this representation was essential for the short proof. (For various other properties of $f_\kappa(z)$, see Polya and Szegö [23, p. 7], Truesdell [28] and the literature given there, Lawden [13], Zeitlin [33], Miesner and Wirsing [19]. For $\kappa < 0$, see also LeRoy [14], Sandham [25], Levin [15].)

Functions of the form F(z) may be regarded as generalizations of polynomials (insofar as the coefficient of the highest power of z is itself a function of z belonging to a certain class), and they have in common with polynomials the property of admitting at most k zeros in the complex plane with a cut along the real axis from 1 to ∞ (Theorem 1). (Various other properties of functions F(z) have been investigated by Hadamard [5], LeRoy [14] (regularity); Wall [30], [31], Thale [27], Merkes [17], Seall and Wetzel [26] ($k = 0$ and continued fractions); Climescu [4] (characterization of a case similar to $k = 2$); Kaplan [8] (real zeros of entire functions); Marden [16] (zeros); Bendat and Sherman [1], Korányi [9], [10] (characterization of a case similar to $k = 1$).)

The maximum number of zeros is not attained in all cases; in Theorems 2 and 3 we give two instances where the maximum number is actually reached. Theorem 3 contains a refinement of a special case of the Borel-Laguerre Theorem; we shall use it in Section 5 to give a new proof of Hurwitz's theorem on the complex zeros of Bessel functions of order less than -1.

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Kuttner [12] has recently introduced the function

$$g_\kappa(z) = \sum_0^\infty (1 - c^{n+1})^\kappa z^n \qquad (0 < c < 1,\ \kappa > 0);$$

like $f_\kappa(z)$, it plays a role in Riesz summability. We show in Theorems 4 and 5 that both functions have real zeros only. Both functions have exactly k simple zeros on the negative real axis; we also obtain some further information on the location of these zeros.

1. AN UPPER BOUND FOR THE NUMBER OF ZEROS

In this section we shall show that functions of the type

$$f(z) = P_{k-1}(z) + z^k \int_0^1 \frac{dg(t)}{1 - zt},$$

where $P_{k-1}(z)$ is a real polynomial of degree at most $k - 1$ $(k = 1, 2, \cdots;$ $P_{-1}(z) = 0)$, have at most k zeros in the set

$$C^* = \{x + iy \mid x < 1 \text{ if } y = 0\}$$

if $g(t)$ increases (unless $f(z) \equiv 0$).

Numbers denoted by z, ζ, or ζ_i always belong to C^*, and $g(t)\uparrow$ means that $g(t)$ is increasing in the wider sense. $V(a, b)$ denotes the space of functions of bounded variation in $[a, b]$. Our first lemma is substantially due to LeRoy [14, pp. 330-331].

LEMMA 1. *If*

$$f(z) = \int_0^1 \frac{dg(t)}{1 - zt} \qquad (g(t)\uparrow,\ g(1) > g(0)),$$

then $f(z) \neq 0$ *for* $z \in C^*$.

Proof. If $z = re^{i\theta} \in C^*$, then

$$f(z) = \int_0^1 \frac{1 - rt\cos\theta}{|1 - zt|^2}\, dg(t) + ir\sin\theta \int_0^1 \frac{t}{|1 - zt|^2}\, dg(t).$$

We may assume that $g(1) > g(+0)$ (otherwise $f(z) = g(+0) - g(0) > 0$), and this implies that the second integral is positive. If $\cos\theta = 1$ (in this case, $0 \leq r < 1$) or $\cos\theta = -1$, then $1 - rt\cos\theta > 0$, and this implies that the first integral is also positive; this proves the lemma.

LEMMA 2. *If*

$$f(z) = A + z\int_0^1 \frac{dg(t)}{1 - zt} \qquad (g(t)\uparrow,\ g(1) > g(0),\ A \textit{ real}),$$

then all zeros of $f(z)$ $(z \in C^*)$ *are real.*

Proof. If $z = re^{i\theta} \in C^*$, then

$$\Im f(z) = r \sin \theta \int_0^1 \frac{dg(t)}{|1 - zt|^2},$$

and this integral is positive (we denote the real and imaginary parts of a complex number a by $\Re a$ and $\Im a$, respectively).

LEMMA 3. *Let*

$$f(z) = \sum_{\nu=0}^{k-1} A_\nu z^\nu + z^k \int_0^1 \frac{dg(t)}{1 - zt} \qquad (k = 1, 2, \cdots, g(t) \in V(0, 1));$$

then, for $\zeta \in C^*$ *and* $\zeta \neq 0$,

$$\frac{f(z)}{1 - z/\zeta} = \left(\frac{z}{\zeta}\right)^{k-1} \frac{f(\zeta)}{1 - z/\zeta} + \sum_{n=0}^{k-2} z^n \sum_{\nu+\mu=n} A_\nu \zeta^{-\mu} - \zeta z^{k-1} \int_0^1 \frac{dg(t)}{(1 - zt)(1 - \zeta t)}$$

(empty sums are zero).

Proof. If $|z|$ is small, then, with the notation $A_{\nu+k} = \int_0^1 t^\nu dg(t)$ $(\nu = 0, 1, \cdots)$,

$$\frac{f(z)}{1 - z/\zeta} = \frac{1}{1 - z/\zeta} \sum_{\nu=0}^{\infty} A_\nu z^\nu = \sum_{n=0}^{\infty} z^n \sum_{\nu+\mu=n} A_\nu \zeta^{-\mu};$$

for $n \geq k - 1$,

$$\sum_{\nu+\mu=n} A_\nu \zeta^{-\mu} = \sum_{\nu=0}^{k-1} A_\nu \zeta^{\nu-n} + \zeta^{-n} \sum_{\nu=k}^{n} \int_0^1 (t\zeta)^{\nu-k} \zeta^k dg(t)$$

$$= \zeta^{-n}\left(f(\zeta) - \zeta^k \int_0^1 \frac{dg(t)}{1 - t\zeta} + \zeta^k \int_0^1 \frac{1 - (t\zeta)^{n-k+1}}{1 - t\zeta} dg(t)\right)$$

$$= \zeta^{-n} f(\zeta) - \zeta \int_0^1 \frac{t^{n-k+1}}{1 - t\zeta} dg(t).$$

The result follows from a short calculation.

Let $f(z)$ be defined as in Lemma 3, and assume that $f(\zeta_1) = \cdots = f(\zeta_p) = 0$ and $f(z) \prod_{i=1}^{p} (1 - z/\zeta_i)^{-1}$ is regular; then, by a repeated application of Lemma 3 for $p \leq k$, we see that

$$f(z) \prod_{i=1}^{p} (1 - z/\zeta_i)^{-1} = B_0 + B_1 z + \cdots + B_{k-p-1} z^{k-p-1}$$

$$+ (-1)^p \zeta_1 \cdots \zeta_p z^{k-p} \int_0^1 \frac{dg(t)}{(1 - \zeta_1 t) \cdots (1 - \zeta_p t)(1 - zt)}$$

for some $B_0, \cdots, B_{k-p-1}$ (all B_j are 0, for $p = k$), and $B_0 = A_0$ for $p < k$. It follows, in particular, that

$$(1) \quad f(z) \prod_{i=1}^{k-1} \left(1 - \frac{z}{\zeta_i}\right)^{-1} = A_0 + (-1)^{k-1} \zeta_1 \cdots \zeta_{k-1} z \int_0^1 \frac{dg(t)}{(1 - \zeta_1 t) \cdots (1 - \zeta_{k-1} t)(1 - zt)},$$

$$(2) \quad f(z) \prod_{i=1}^{k} \left(1 - \frac{z}{\zeta_i}\right)^{-1} = (-1)^k \zeta_1 \cdots \zeta_k \int_0^1 \frac{dg(t)}{(1 - \zeta_1 t) \cdots (1 - \zeta_k t)(1 - zt)}.$$

THEOREM 1. *Let*

$$f(z) = \sum_{\nu=0}^{k-1} A_\nu z^\nu + z^k \int_0^1 \frac{dg(t)}{1 - zt} \qquad (g(t) \uparrow, \ A_0, \cdots, A_{k-1} \ \textit{real});$$

then $f(z)$ *has at most* k *zeros in* C^*, *unless* $f(z) \equiv 0$.

Proof. Because of Lemma 1 we may assume that $k \geq 1$; we also assume that $g(1) > g(0)$ and $f(0) \neq 0$. If $f(\zeta) = 0$, then also $f(\bar{\zeta}) = 0$ (since the A_i are real), and we assume that $f(z)$ has more than k zeros in C^*. If k is even, select zeros $\zeta_1, \cdots, \zeta_k$ in such a way that with ζ_i, $\bar{\zeta}_i$ also appears in this sequence. It follows from (2) that

$$f(z) = \prod_{i=1}^{k} (z - \zeta_i) \int_0^1 \frac{1}{1 - zt} d \int_0^t \frac{dg(\tau)}{(1 - \zeta_1 \tau) \cdots (1 - \zeta_k \tau)},$$

and Lemma 1 shows that $f(z)$ has no other zero (since $(1 - \zeta_1 \tau) \cdots (1 - \zeta_k \tau) > 0$ for $0 \leq \tau \leq 1$).

If k is odd, select zeros $\zeta_1, \cdots, \zeta_{k-1}$ in such a way that with ζ_i, $\bar{\zeta}_i$ also appears in this sequence. It follows from (1) that

$$f(z) = \prod_{i=1}^{k-1} \left(1 - \frac{z}{\zeta_i}\right) \left(A_0 + (-1)^{k-1} \zeta_1 \cdots \zeta_{k-1} z \int_0^1 \frac{dg(t)}{(1 - \zeta_1 t) \cdots (1 - \zeta_{k-1} t)(1 - zt)}\right),$$

and any other zero of $f(z)$ is real, by Lemma 2. The result now follows again from (2) and Lemma 1. If $z = 0$ is a q-fold zero ($1 \leq q$, and obviously $q \leq k$), then we replace the function $f(z)$ in this proof by

$$z^{-q} f(z) = \sum_{\nu=q}^{k-1} A_\nu z^{\nu-q} + z^{k-q} \int_0^1 \frac{dg(t)}{1 - zt}.$$

Remarks. (1) For $|z| < 1$, the function $f(z)$ in Theorem 1 admits the expansion

$$f(z) = \sum_{\nu=0}^{k-1} A_\nu z^\nu + \sum_{\nu=k}^{\infty} z^\nu \int_0^1 t^{\nu-k} dg(t),$$

and it follows from the Hausdorff moment problem (Hausdorff [6]) that a power series $\sum_0^\infty a_n z^n$ is of this type if $\Delta^m a_n \geq 0$ ($m = 0, 1, 2, \cdots$; $n = k, k+1, \cdots$; $\Delta^0 a_n = a_n$, $\Delta a_n = a_n - a_{n+1}$, $\Delta^{m+1} a_n = \Delta(\Delta^m a_n)$). This criterion will be used in later applications.

(2) A function

$$h(z) = \sum_{\nu=0}^{k-1} A_\nu z^\nu + z^p \int_0^1 \frac{dg(t)}{1 - zt}$$

$$(0 \leq p < k, \ g(t) \uparrow, \ g(1) > g(+0), \ A_0, \cdots, A_{k-1} \text{ real})$$

can also be written in the form

$$\sum_{\nu=0}^{k-1} B_\nu z^\nu + z^k \int_0^1 \frac{d\gamma(t)}{1 - zt} \qquad (\gamma(t) \uparrow, \ \gamma(1) > \gamma(0), \ B_0, \cdots, B_{k-1} \text{ real}).$$

If we write $\gamma(t) = \int_0^t t^{k-p} dg(t)$, then this is easily seen from the relation

$$z^p \int_0^1 \frac{dg(t)}{1 - zt} - z^k \int_0^1 \frac{t^{k-p}}{1 - zt} dg(t) = z^p \sum_{\nu=0}^{k-p-1} z^\nu \int_0^1 t^\nu dg(t).$$

Marden [16] has shown that functions $h(z)$ with $p = 0$ and with $\int_0^1 \frac{dg(t)}{1 - zt}$ meromorphic have at most k zeros in the region

$$z = 1 + re^{i\theta} \qquad \left(\frac{k}{k+1}\pi < \theta < \frac{k+2}{k+1}\pi\right).$$

(It should be noted that Marden's theorem also deals with the case where $\int_0^1 \frac{dg(t)}{1 - zt}$ has complex residues.)

2. THE EXACT NUMBER OF ZEROS

The function $f(z)$ in Theorem 1 may have fewer than k zeros (even if $g(1) > g(0)$). This is shown by the example

$$f(z) = 1 + z \int_0^1 \frac{tdt}{1 - zt} = -\frac{1}{z} \log(1 - z).$$

The following theorems give some cases where the maximum number of zeros is attained.

THEOREM 2. *Let* $g(t) \in V(0, 1)$ *be a function with the following properties:*

(i) *there exists an* α $(1 \leq \alpha < 2)$ *such that for every* $0 < \delta < 1$

$$\inf_{\delta \leq t < t' < 1} \frac{g(t') - g(t)}{(t' - t)^{\alpha}} = \eta(\delta) > 0,$$

(ii) $r \int_0^1 \frac{dg(t)}{1 + rt} \to \infty \quad (r \to \infty), \qquad \int_0^1 \frac{dg(t)}{1 - t + \rho t} \to \infty \quad (0 < \rho \to 0).$

Then the function

$$f(z) = \sum_{\nu=0}^{k-1} A_\nu z^\nu + z^k \int_0^1 \frac{dg(t)}{1 - zt} \qquad (z \in C^*,\ k \geq 1,\ A_0, \cdots, A_{k-1}\ \textit{real})$$

has exactly k *zeros in* C^*.

Proof. Let $f_\tau(z) = \tau \sum_{\nu=0}^{k-1} A_\nu z^\nu + z^k \int_0^1 \frac{dg(t)}{1 - zt}$, $0 \leq \tau \leq 1$. For $z = re^{i\theta} \in C^*$ we have the formulas (see Lemma 1 and Lemma 2)

$$\Im \frac{f_\tau(z)}{z^{k-1}} = -\tau \sum_{m=1}^{k-1} A_{k-1-m} \frac{\sin m\theta}{r^m} + r \sin\theta \int_0^1 \frac{dg(t)}{|1 - zt|^2},$$

$$\Re \frac{f_\tau(z)}{z^k} = \tau \sum_{m=1}^{k} A_{k-m} \frac{\cos m\theta}{r^m} + \int_0^1 \frac{1 - rt\cos\theta}{|1 - zt|^2}\, dg(t);$$

it follows that

$$|f_\tau(z)| \geq |z|^{k-1} |\sin\theta| \left(r \int_0^1 \frac{dg(t)}{|1 - zt|^2} - \frac{C_1}{r} \right) \tag{3}$$

for some $C_1 > 0$ that is independent of r, θ, and τ, as long as $r \geq 1$, $|\theta| \leq 3\pi/4$, and $0 \leq \tau \leq 1$. Also,

$$|f_\tau(z)| \geq |z|^k \left(\int_0^1 \frac{1 - rt\cos\theta}{|1 - zt|^2}\, dg(t) - \frac{C_2}{r} \right) \tag{4}$$

for some $C_2 > 0$ that is independent of r and τ, as long as $r \geq 1$, $0 \leq \tau \leq 1$. For $r \geq 1$ and $0 < \theta < \frac{\pi}{2}$ $\left(\text{observe that } \sin\theta + \cos\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) > 1\right)$,

$$r \int_0^1 \frac{dg(t)}{|1 - zt|^2} \geq r \int_{1/r(\sin\theta + \cos\theta)}^{1/r} \frac{dg(t)}{(1 - rt\cos\theta)^2 + (rt\sin\theta)^2}$$

$$\geq \frac{r}{2\sin^2\theta}\left[g\left(\frac{1}{r}\right) - g\left(\frac{1}{r(\sin\theta+\cos\theta)}\right)\right]$$

$$\geq \frac{r}{2\sin^2\theta}\,\eta\left(\frac{1}{r\sqrt{2}}\right) r^{-\alpha}\left(\frac{\sin\left(\theta+\frac{\pi}{4}\right) - \sin\frac{\pi}{4}}{\sin\left(\theta+\frac{\pi}{4}\right)}\right)^{\alpha}$$

$$= \frac{1}{2}\,\eta\left(\frac{1}{r\sqrt{2}}\right)\frac{r^{1-\alpha}}{\sin^2\theta}\left(\frac{\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\sin\theta}{\sin\left(\theta+\frac{\pi}{4}\right)\cos\frac{\theta}{2}}\right)^{\alpha}$$

$$\geq \frac{1}{2}\,\eta\left(\frac{1}{r\sqrt{2}}\right)\frac{(\sin\theta)^{\alpha-2}}{r^{\alpha-1}}\left(\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{\alpha}$$

From this estimate and (3) it follows that

$$(5)\qquad |f_\tau(z)| \geq r^{k-1}|\sin\theta|\left(\frac{1}{2}\,\eta\left(\frac{1}{r\sqrt{2}}\right)\frac{|\sin\theta|^{\alpha-2}}{r^{\alpha-1}}\left[\cos\left(\frac{|\theta|}{2}+\frac{\pi}{4}\right)\right]^{\alpha} - \frac{C_1}{r}\right)$$

if $r \geq 1$, $0 < |\theta| < \frac{\pi}{2}$, $0 \leq \tau \leq 1$. If $r \geq 1$ and $0 \leq \theta \leq \frac{3}{4}\pi$, then

$$r\int_0^1 \frac{dg(t)}{|1-zt|^2} \geq r\int_0^1 \frac{dg(t)}{(1+rt)^2} \geq \frac{1}{2}\int_0^1 \frac{dg(t)}{1+rt},$$

and it follows that

$$(6)\qquad |f_\tau(z)| \geq r^{k-2}|\sin\theta|\left(\frac{r}{2}\int_0^1 \frac{dg(t)}{1+rt} - C_1\right)$$

if $r \geq 1$, $|\theta| \leq \frac{3}{4}\pi$, $0 \leq \tau \leq 1$. For $\frac{3}{4}\pi \leq \theta \leq \frac{5}{4}\pi$,

$$\int_0^1 \frac{1 - rt\cos\theta}{|1-zt|^2}\,dg(t) \geq \frac{1}{\sqrt{2}}\int_0^1 \frac{1+rt}{(1+rt)^2}\,dg(t),$$

and it follows from (4) that

$$(7)\qquad |f_\tau(z)| \geq r^{k-1}\left(\frac{r}{\sqrt{2}}\int_0^1 \frac{dg(t)}{1+rt} - C_2\right)$$

if $r \geq 1$, $\frac{3}{4}\pi \leq \theta \leq \frac{5}{4}\pi$, $0 \leq \tau \leq 1$.

We next investigate the integral $\int_0^1 \frac{dg(t)}{1-zt}$ in a neighborhood of $z = 1$. Let $z = 1 + \rho e^{i\phi}$ $\left(0 < \rho \leq \frac{1}{2},\ \frac{\pi}{2} \leq \phi \leq \pi\right)$. We have the relation

$$\int_0^1 \frac{dg(t)}{1 - zt} = \int_0^1 \frac{1 - t - \rho t \cos \phi}{(1 - t)^2 - 2\rho t(1 - t)\cos \phi + (\rho t)^2}\, dg(t)$$

$$+ i \int_0^1 \frac{\rho t \sin \phi}{(1 - t)^2 - 2\rho t(1 - t)\cos \phi + (\rho t)^2}\, dg(t),$$

and since $|a + ib| \geq \frac{1}{2}(|a| + |b|)$,

$$\left|\int_0^1 \frac{dg(t)}{(1 - zt)}\right| \geq \frac{1}{2} \int_0^1 \frac{1 - t + \rho t(\sin \phi - \cos \phi)}{(1 - t)^2 - 2\rho t(1 - t)\cos \phi + (\rho t)^2}\, dg(t)$$

$$= \frac{1}{2} \int_0^1 \frac{1 - t + \sqrt{2}\rho t \sin(\phi - (\pi/4))}{(1 - t)^2 - 2\rho t(1 - t)\cos \phi + (\rho t)^2}\, dg(t) \geq \frac{1}{2} \int_0^1 \frac{1 - t + \rho t}{(1 - t + \rho t)^2}\, dg(t).$$

It follows that

$$|f_\tau(z)| \geq |z|^k \left(\frac{1}{2} \int_0^1 \frac{dg(t)}{(1 - t + \rho t)} - C_3 \right) \tag{8}$$

for some $C_3 > 0$ that is independent of ρ, ϕ, and τ, as long as $\rho \leq \frac{1}{2}$, $\frac{\pi}{2} \leq \phi \leq \frac{3\pi}{2}$, in $z = 1 + \rho e^{i\phi}$, and $0 \leq \tau \leq 1$.

Since $g(t)$ has the property (ii), we can choose an $R > 1$ such that

$$\frac{R}{2} \int_0^1 \frac{dg(t)}{1 + Rt} \geq 2C_1, \quad \frac{R}{\sqrt{2}} \int_0^1 \frac{dg(t)}{1 + Rt} \geq 2C_2 .$$

Next we choose θ_0 with $0 < \theta_0 < \frac{\pi}{4}$ such that

$$\frac{1}{2}\, \eta\!\left(\frac{1}{r\sqrt{2}}\right) \frac{|\sin \theta_0|^{\alpha - 2}}{r^{\alpha - 1}} \geq \frac{2C_1}{r} \left(\cos \frac{3}{8}\pi\right)^{-\alpha}$$

for $1 \leq r \leq R$ (this is possible because of (i) and because $\alpha < 2$; observe that $\eta(x)$ is an increasing function of x) and such that

$$\frac{1}{2} \int_0^1 \frac{dg(t)}{(1 - t + \rho t)} \geq 2C_3 \qquad (\rho = \tan \theta_0,\ \rho < 1/2).$$

It follows from (5), (6), (7), and (8) that

$$|f_\tau(z)| \geq R^{k-2} |\sin \theta_0|\, C_1 \text{ for } z = Re^{i\theta} \qquad \left(\theta_0 \leq |\theta| \leq \frac{3}{4}\pi\right), \tag{9}$$

$$|f_\tau(z)| \geq R^{k-1} C_2 \text{ for } z = Re^{i\theta} \qquad \left(\frac{3}{4}\pi \leq \theta \leq \frac{5}{4}\pi\right), \tag{10}$$

(11) $\quad |f_\tau(z)| \geq r^{k-2}|\sin\theta_0|\, C_1 \text{ for } z = re^{\pm i\theta_0} \quad (1 \leq r \leq R),$

(12) $\quad |f_\tau(z)| \geq |z|^k C_3 \text{ for } z = 1 + e^{i\phi}\tan\theta_0 \quad \left(\frac{\pi}{2} \leq \phi \leq \frac{3}{2}\pi\right).$

From (9), (10), (11), and (12) we see that there exists a closed curve C, containing the origin in its interior, such that $f_\tau(z) \neq 0$ for $z \in C$ and $0 \leq \tau \leq 1$. By virtue of the latter property, $f_0(z)$ and $f_1(z)$ have the same number of zeros inside of C. By Lemma 1, $f_0(z)$ has exactly k zeros inside of C. Theorem 2 now follows from Theorem 1.

Remarks. 1. The first condition in (ii) is satisfied if

$$g(+0) > g(0) \quad \text{or} \quad \int_{1/r}^{1} \frac{dg(t)}{t} \to \infty \quad (r \to \infty).$$

The second condition in (ii) is satisfied if

$$g(1) > g(1-0) \quad \text{or} \quad \int_0^{\delta} \frac{dg(t)}{1-t} \to \infty \quad \delta \uparrow 1.$$

For $\alpha = 1$, a simple calculation shows that the second condition in (ii) also follows from (i). Condition (i) is satisfied for $\alpha = 1$ if $g'(t)$ exists and

$$\inf_{\delta \leq t \leq 1} g'(t) = \eta(\delta) > 0.$$

2. For $\tau = 0$ the function

$$f_\tau(z) = \tau \sum_{\nu=0}^{k-1} A_\nu z^\nu + z^k \int_0^1 \frac{dg(t)}{1-zt}$$

has k zeros at the origin. The proof of Theorem 2 shows that these zeros cannot move to ∞ or to a point on the real axis to the right of $z = 1$, as τ moves from 0 to 1. It turns out that the first condition in (ii) prevents movement to ∞; the second condition in (ii) prevents an approach to $z = 1$, while condition (i) prevents a zero from approaching the real axis to the right of $z = 1$. Theorem 2 is no longer true for a step-function g(t), even if (ii) is satisfied. We mention the following example:

$$f(z) = 2 - z + z^2 \int_0^1 \frac{dg(t)}{1-zt}, \quad g(t) = \begin{cases} 0 & (t = 0), \\ \frac{3}{16} & (0 < t < 1), \\ \frac{3}{8} & (t = 1), \end{cases}$$

that is, $16(1-z)f(z) = (2-z)(3z^2 - 16z + 16)$; here f(z) has the zeros 4, 2, 4/3 (by Remark 1, condition (ii) is satisfied). In order to obtain a result similar to that of Theorem 2 for step functions, we must find another condition instead of (i), whereas (ii) (that is, (6), (7), and (8)) can again be used. The following theorem is of this type. We use the notation $C^*_\alpha = \{x + iy \mid \text{ if } x \geq \alpha, \text{ then } y \neq 0\}$.

THEOREM 3. *Let* $f(z)$ *be a real entire function of order* $\rho < 1$ *with infinitely many zeros* $\{a_i\}$ $(i = 1, 2, \cdots)$, *and assume that for some nonnegative integer* k,

(a) $a_i \in C^*_0$ $(i = 1, 2, \cdots, k)$,

(b) $0 < a_{k+1} < a_{k+2} < \cdots$.

Let $g(z) = Af(z) + zf'(z)$ (A *real*), *and assume that* $g(z)$ *has exactly one real zero between two zeros* a_ν, $a_{\nu+1}$ $(\nu = k+1, k+2, \cdots)$. *Then* $g(z)$ *has exactly* $k + 1$ *zeros in* $C^*_{a_{k+1}}$. (We remark that all zeros of $f(z)$ are real if $k = 0$; the zeros in (b) are all simple.)

Proof. If we write $F(z) = f(a_{k+1} z)$, $G(z) = g(a_{k+1} z)$, then $G(z) = AF(z) + zF'(z)$, and this shows that we may assume that $a_{k+1} = 1$. We write

$$f(z) = \alpha P(z) \prod_{\nu=k+1}^{\infty} \left(1 - \frac{z}{a_\nu}\right) = \alpha P(z) h(z),$$

where $P(z)$ is a real polynomial of degree k and $P(z) > 0$ for $z \geq 0$. It follows that

$$g(z) = \alpha h(z) \left(AP(z) + z P'(z) + z P(z) \frac{h'(z)}{h(z)}\right)$$

and (for $|z|$ small)

$$z \frac{h'(z)}{h(z)} = - \sum_{m=1}^{\infty} z^m \sum_{\nu=k+1}^{\infty} \frac{1}{a_\nu^m} = - \sum_{m=1}^{\infty} z^m \int_0^1 t^{m-1} dg(t),$$

$$g(t) = \sum_{\substack{a_\nu t \geq 1 \\ \nu \geq k+1}} \frac{1}{a_\nu} \in V(0, 1) \quad \text{because} \quad \sum_{\nu=k+1}^{\infty} a_\nu^{-1} < \infty.$$

(For a similar reasoning, see Kaplan [8].)

Let $P(z) = \sum_{\nu=0}^{k} b_\nu z^\nu$ $(b_k > 0)$; then

$$z P(z) \frac{h'(z)}{h(z)} = - \sum_{n=1}^{k} z^n \int_0^1 \sum_{\substack{\nu+\mu=n \\ \mu \geq 1}} t^{\mu-1} b_\nu \, dg(t) - \sum_{n=k+1}^{\infty} z^n \int_0^1 t^{n-k-1} t^k P\left(\frac{1}{t}\right) dg(t),$$

and it follows that

$$g(z) = \alpha h(z) \left(Q(z) - z^{k+1} \int_0^1 \frac{d\gamma(t)}{1 - zt} \right),$$

where $Q(z)$ is of degree at most k, and where $\gamma(t) = \int_0^t \tau^k P\left(\frac{1}{\tau}\right) dg(\tau) \uparrow$. By Theorem 1, the function $g(z)$ has at most $k + 1$ zeros in C^*.

We next observe that for $n > k+1$ and $n \to \infty$,

$$(13) \qquad \int_{1/a_n}^{1} \frac{d\gamma(t)}{t} = \int_{1/a_n}^{1} t^{k-1} P\left(\frac{1}{t}\right) dg(t) = \sum_{\nu=k+1}^{n-1} \frac{P(a_\nu)}{a_\nu^k} \to \infty,$$

since $\frac{P(a_\nu)}{a_\nu^k} \to b_k > 0$ for $\nu \to \infty$. The function

$$\Phi(z) = z^{k+1} \int_0^1 \frac{d\gamma(t)}{1 - zt} = R(z) - z\,P(z)\frac{h'(z)}{h(z)}$$

($R(z)$ a polynomial of degree at most k) has simple poles for a_{k+1} $(=1)$, $a_{k+2}, \cdots$ with residues $-a_i P(a_i) < 0$ $(i = k+1, k+2, \cdots)$.

We consider next the function

$$F_\tau(z) = \tau Q(z) - z^{k+1} \int_0^1 \frac{d\gamma(t)}{1 - zt}.$$

Since the residues of $\Phi(z)$ are negative, the function $F_\tau(z)$ has at least one zero in every (open) interval (a_i, a_{i+1}) $(i = k+1, k+2, \cdots)$. The function $F_0(z)$ has $k+1$ zeros at the origin. The function $F_\tau(z)$ satisfies inequalities of the type (6) and (7) in the proof of Theorem 2 (where k has to be replaced by $k+1$), and because of (13) and the first remark after Theorem 2, we may choose n in such a way that for $a_n < r < a_{n+1}$

$$\frac{r}{2} \int_0^1 \frac{d\gamma(t)}{1+rt} \geq 2C_1, \quad \frac{r}{\sqrt{2}} \int_0^1 \frac{d\gamma(t)}{1+rt} \geq 2C_2.$$

This implies that

$$(14) \qquad |F_\tau(re^{i\phi})| \geq C_1 r^{k-1} |\sin\theta| \qquad \text{if } |\theta| \leq \tfrac{3}{4}\pi,\ 0 \leq \tau \leq 1,$$

$$(15) \qquad |F_\tau(re^{i\phi})| \geq C_2 r^k \qquad \text{if } \tfrac{3}{4}\pi \leq \theta \leq \tfrac{5}{4}\pi,\ 0 \leq \tau \leq 1.$$

Let $\delta > 0$ be so small that $F_\tau(z) \neq 0$ for $a_n < z \leq a_n + \delta < a_{n+1}$ $(0 \leq \tau \leq 1)$. It follows by (14) and (15) that $F_\tau(z) \neq 0$ for $|z| = a_n + \delta$ and $0 \leq \tau \leq 1$. Therefore, the functions $F_0(z)$ and $F_1(z)$ have the same number of zeros for $|z| \leq a_n + \delta$ (they have the same number of poles). But $F_0(z)$ has at least $k+1+(n-k-1) = n$ zeros in $|z| \leq a_n + \delta$, and this is also true for $F_1(z) = g(z)/\alpha h(z)$. But $g(z)$ has exactly $n-k-1$ zeros between a_{k+1} and $a_n + \delta$, which implies that $g(z)$ has at least $k+1$ zeros in C^*. This proves the theorem.

Remarks. 1. In the proof of Theorem 3 we made use of the inequalities (6) and (7) in the proof of Theorem 2. The first condition in (ii) of Theorem 2 follows from the assumption that $f(z)$ has infinitely many real zeros. Condition (i) in Theorem 2 is replaced with the assumption that the zeros of $f(z)$ and $g(z)$ interlace in a certain way.

2. Theorem 3 generalizes a special case of the theorem of Borel and Laguerre (Borel [2, p. 37]; see also Marden [16]). For theorems of a related structure, see Veržbinskiĭ [29].

3. It is possible to generalize Theorem 3 so that it will give information on the zeros of functions of the form $g(z) = P(z)f(z) + Q(z)f'(z)$, where $P(z)$ and $Q(z)$ are polynomials and $f(z)$ is an entire function of finite order. Here we shall not pursue this question further; the present form of Theorem 3 is sufficient for an application in Section 5.

3. THE FUNCTIONS $f_\kappa(z) = \sum_0^\infty (n+1)^\kappa z^n$ $(\kappa > 0)$

The location of the zeros of $f_\kappa(z)$ plays a role in Riesz summability. The most recent result of Miesner and Wirsing [19] states that $f_\kappa(z)$ (which obviously has an analytic extension into C^*) possesses exactly k zeros in $|z| < 1$ if $2k < \kappa < 2k+2$ $(k = 0, 1, \cdots)$, and that these zeros are negative. Furthermore, $f_{2n}(-1) = 0$ $(n = 1, 2, \cdots)$, and $f_\kappa(e^{i\phi}) \neq 0$ $(0 < \phi < 2\pi)$ in all other cases (for $\kappa \leq 0$ it has been shown by LeRoy [14] that $f_\kappa(z) \neq 0$ for $z \in C^*$).

THEOREM 4. *The zeros of* $f_\kappa(z)$ *in* C^* *are all negative and simple. If* $k < \kappa \leq k+1$, *they are exactly* k *in number, and for* $k = 2n$ *and* $k = 2n+1$, *exactly* n *of them lie in the interval* $-1 < x < 0$. *Moreover,* $f_\kappa(-1) = 0$ *if and only if* $\kappa = 2, 4, 6, \cdots$.

Proof. First we prove, by an application of Theorem 1, that $f_\kappa(z)$ has at most k zeros. For $|z| < 1$,

$$(1-z)^{k+1} f_\kappa(z) = \sum_{n=0}^{\infty} z^n \sum_{\nu+\mu=n} \binom{k+1}{\nu} (-1)^\nu (\mu+1)^\kappa \qquad (k < \kappa \leq k+1),$$

and for $n \geq k$,

$$a_n = \sum_{\nu=0}^{k+1} \binom{k+1}{\nu} (-1)^{\nu+k+1} (n-k+\nu)^\kappa = (-1)^{k+1} \Delta^{k+1} (n-k)^\kappa .$$

(For this notation and the following reasoning, see Remark 1, after Theorem 1.)

But for $n \geq k$,

$$\begin{aligned} \Delta^p a_n &= (-1)^{k+1} \Delta^{k+p+1} (n-k)^\kappa \\ &= (-1)^{k+1} (-1)^{k+p+1} \kappa(\kappa-1) \cdots (\kappa-k-p) [n-k+\eta(k+p+1)]^{\kappa-(k+p+1)} \end{aligned}$$

for some $0 < \eta < 1$ (see for example Nörlund [20]). It follows that $\Delta^p a_n \geq 0$ $(n \geq k,\ p = 0, 1, \cdots)$ and

$$(1-z)^{k+1} f_\kappa(z) = \sum_{n=0}^{k-1} z^n \sum_{\nu+\mu=n} \binom{k+1}{\nu} (-1)^\nu (\mu+1)^\kappa + z^k \int_0^1 \frac{dg(t)}{1-zt}$$

for some increasing $g(t)$. By Theorem 1, this representation implies that $f_\kappa(z)$ has at most k zeros.

We show next that $xf_\kappa(x) \to 0$ as $x \to -\infty$. From the formula

$$a_n = (-1)^{k+1} \Delta^{k+1} (n-k)^\kappa = (k+1)! \qquad (\kappa = k+1,\ n \geq k)$$

it follows that

$$f_{k+1}(z) = \frac{P_k(z)}{(1-z)^{k+2}},$$

where $P_k(z)$ denotes a polynomial of degree k (Pólya and Szegö [23], Truesdell [28], Lawden [13], Zeitlin [33]). From this representation and the relation

$$(n+1)^{-\theta} = \frac{1}{\Gamma(\theta)} \int_0^1 \left(\log \frac{1}{t}\right)^{\theta - 1} t^n \, dt \qquad (\theta > 0)$$

we have, for $\kappa = k + 1 - \theta$, the formula

$$f_\kappa(z) = \frac{1}{\Gamma(\theta)} \int_0^1 \left(\log \frac{1}{t}\right)^{\theta-1} \frac{P_k(zt)}{(1-zt)^{k+2}} \, dt,$$

and it follows that for $x > 1$ and $0 < \theta < 1$,

$$-xf_\kappa(-x) = O(x) \int_0^{1/x} \left(\log \frac{1}{t}\right)^{\theta-1} dt + O\left(\frac{1}{x}\right) \int_{1/x}^{1/\sqrt{x}} \left(\log \frac{1}{t}\right)^{\theta-1} \frac{dt}{t^2}$$

$$+ O\left(\frac{1}{x}\right) \int_{1/\sqrt{x}}^{1} \left(\log \frac{1}{t}\right)^{\theta-1} \frac{dt}{t^2}$$

$$= O(1)(\log x)^{\theta-1} + O(1)(\log \sqrt{x})^{\theta-1} + O\left(\frac{1}{\sqrt{x}} (\log \sqrt{x})^\theta\right) = O(1)(\log x)^{\theta-1}.$$

Obviously,

$$-xf_{k+1}(-x) = O\left(\frac{1}{x}\right),$$

which proves that $xf_\kappa(x) \to 0$ as $x \to -\infty$. We shall now use this result to show that $f_\kappa(z)$ has exactly k negative zeros. The proof is by induction, and it is based on the relation $(zf_\kappa)' = f_{\kappa+1}(z)$. We also use the fact that $f_\kappa(-1) = (1 - 2^{1+\kappa})\, \zeta(-\kappa)$, which implies that $f_\kappa(-1) = 0$ only if $\kappa = 2, 4, 6, \cdots$ (here ζ denotes Riemann's ζ-function; see for example, Chapman [3]). If $f_\kappa(z)$ has k negative and simple zeros for $k < \kappa \leq k+1$ (which is true for $k = 0$), then it follows from Rolle's theorem that $f_{\kappa+1}(z) = (zf_\kappa(z))'$ has at least $k+1$ negative (and distinct) zeros, and this is the maximum number of zeros for $f_{\kappa+1}(z)$. Furthermore, if $2n < \kappa < 2n+2$ $(n = 0, 1, 2, \cdots)$, then the integral

$$I_\kappa = \int_C \frac{f_\kappa'(z)}{f_\kappa(z)} \, dz,$$

where C consists of the curves

$$z = it \quad (-1 \le t \le +1) \qquad \text{and} \qquad z = e^{i\phi} \quad \left(\frac{\pi}{2} \le \phi \le \frac{3}{2}\pi\right),$$

is constant, since all the zeros of $f_\kappa(z)$ are real and $f_\kappa(-1) \neq 0$, $f_\kappa(0) = 1$. But $P_{2n}(z) = z^{2n} P_{2n}(1/z)$; that is, P_{2n} is reciprocal (Jonquière's relation; see Truesdell [28], Lawden [13], Peyerimhoff [21]), and therefore $I_{2n+1} = 2\pi in$. The proof of Theorem 4 is now complete if we observe that $f_{2n}(z)$ has $n - 1$ zeros in the interval $-1 < x \le 0$ $(n = 1, 2, \cdots)$, which follows again from Jonquière's relation.

4. THE FUNCTION $g_\kappa(z) = \sum_0^\infty (1 - c^{n+1})^\kappa z^n$ $(0 < c < 1,\ \kappa > 0)$

In this section I use remarks by Professor R. E. Chamberlin.

The function $g_\kappa(z)$ was recently introduced by Kuttner [12] in connection with some questions of Riesz summability. By expanding $(1 - c^{n+1})^\kappa$ into a binomial series, Kuttner showed that

$$g_\kappa(z) = \sum_{\nu=0}^\infty A_\nu^{-\kappa-1} \frac{1}{c^{-\nu} - z} = \Delta^\kappa \frac{1}{c^{-\nu} - z}\bigg|_{\nu=0} \qquad \text{for } |z| < 1 \quad \left(A_\nu^\alpha = \binom{\nu+\alpha}{\nu}\right),$$

and it follows that $g_\kappa(z)$ is meromorphic $(z \neq 1, c^{-1}, c^{-2}, \cdots)$. If $\kappa = k + \theta$, $k = 0, 1, 2, \cdots$, $0 < \theta \le 1$, then

$$\operatorname{sgn} A_\nu^{-\kappa-1} = (-1)^\nu \quad (\nu \le k+1), \qquad \operatorname{sgn} A_\nu^{-\kappa-1} = (-1)^{k+1} \quad (\nu > k+1,\ 0 < \theta < 1),$$

$$A_\nu^{-\kappa-1} = 0 \quad (\nu > k+1,\ \theta = 1).$$

It follows from the fact that $\operatorname{sgn} A_\nu^{-\kappa-1}$ is constant for $\nu \ge k+1$ that $g_\kappa(z)$ has at least one zero between $c^{-\nu}$ and $c^{-(\nu+1)}$ $(\nu \ge k+1$, κ not an integer). In what follows, we shall investigate the zeros of $g_\kappa(z)$ for $z \in C^*$. For a fixed c, we write

$$\sigma_\nu(t) = \begin{cases} 0 & (0 \le t < c^\nu), \\ c^\nu A_\nu^{-\kappa-1} & (c^\nu \le t \le 1) \end{cases}$$

and

$$\gamma_n = \sum_{\nu=n}^\infty \sigma_\nu(t).$$

Observe that

$$\int_0^1 \frac{|d\gamma_n(t)|}{t} = \sum_{\nu=n}^\infty |A_\nu^{-\kappa-1}| < \infty.$$

LEMMA 4. *For* $\kappa > 0$ *and* $p = 0, 1, 2, \cdots$,

$$g_\kappa(z) \prod_{\nu=0}^{p} (1 - c^\nu z) = P_{p-1}(z) + z^p \int_0^1 \frac{1}{1 - zt} \prod_{\nu=0}^{p} (t - c^\nu) \frac{d\gamma_{p+1}(t)}{t},$$

where $P_p(z)$ *is a real polynomial of degree at most* p $(P_{-1} = 0)$.

Proof. For $p = 0$,

$$(1 - z) g_\kappa(z) = \sum_{n=0}^{\infty} ((1 - c^{n+1})^\kappa - (1 - c^n)^\kappa) z^n = \sum_{n=0}^{\infty} z^n \sum_{\nu=1}^{\infty} A_\nu^{-\kappa-1} c^{n\nu} (c^\nu - 1)$$

$$= \sum_{n=0}^{\infty} z^n \sum_{\nu=1}^{\infty} \int_0^1 t^n (t - 1) \frac{d\sigma_\nu(t)}{t} = \int_0^1 \frac{1}{1 - zt} (t - 1) \frac{d\gamma_1(t)}{t},$$

and by induction,

$$(1 - c^{p+1} z) g_\kappa(z) \prod_{\nu=0}^{p} (1 - c^\nu z) = (1 - c^{p+1} z) P_{p-1}(z) + z^p \int_0^1 \prod_{\nu=0}^{p} (t - c^\nu) \frac{d\gamma_{p+1}(t)}{t}$$

$$+ z^{p+1} \int_0^1 \frac{\prod_{\nu=0}^{p+1} (t - c^\nu)}{1 - zt} \frac{d[\gamma_{p+2}(t) + \sigma_{p+1}(t)]}{t};$$

but

$$\int_0^1 \frac{1}{1 - zt} \prod_{\nu=0}^{p+1} (t - c^\nu) \frac{d\sigma_{p+1}(t)}{t} = \frac{1}{1 - zc^{p+1}} \prod_{\nu=0}^{p+1} (c^{p+1} - c^\nu) A_{p+1}^{-\kappa-1} = 0,$$

which proves the lemma.

It follows from

$$\operatorname{sgn} A_\nu^{-\kappa-1} = (-1)^{k+1} \qquad (\nu \geq k+1,\ A_\nu^{-\kappa-1} = 0 \text{ for } \nu > k+1 \text{ if } \theta = 1)$$

that $(-1)^{k+1} \gamma_{k+1}(t) \uparrow$ for $t \uparrow$. Furthermore, $\gamma_{k+1}(t) = \text{constant}$ for $c^{k+1} \leq t \leq 1$, so that, by Lemma 4 (with $p = k$)

$$g_\kappa(z) \prod_{\nu=0}^{k} (1 - c^\nu z) = P_{k-1}(z) + z^k \int_0^1 \frac{dg(t)}{1 - zt} \tag{16}$$

with

$$g(t) = \int_0^t \prod_{\nu=0}^{k} (\tau - c^\nu) \frac{d\gamma_{k+1}(\tau)}{\tau}$$

$(0 \leq t \leq c^{k+1},\ g(t) = g(c^{k+1}),\ c^{k+1} < t \leq 1)$. Since $\operatorname{sgn} \prod_{\nu=0}^{k} (t - c^\nu) = (-1)^{k+1}$ for $0 \leq t \leq c^{k+1}$, $g(t)$ is an increasing function of t.

It now follows from Theorem 1 that $g_\kappa(z)$ has at most k zeros in C^*.

As another consequence of (16) we obtain information on the behavior of $\operatorname{sgn} g_\kappa(-x)$ $(x > 0)$ as $x \to \infty$. For sufficiently large x,

$$\int_0^1 \frac{dg(t)}{1+xt} \geq \frac{1}{2}\int_0^{1/x} dg(t) = \frac{1}{2}\int_0^{1/x} \prod_{\nu=0}^{k} (t - c^\nu)\frac{d\gamma_{k+1}(t)}{t}$$

$$\geq \frac{1}{2} \sum_{c^\mu \leq 1/x} \prod_{\nu=0}^{k} (c^\nu - c^\mu)\,|A_\mu^{-\kappa-1}|$$

$$\geq \frac{1}{4} \prod_{\nu=0}^{k} c^\nu \sum_{c^\mu \leq 1/x} |A_\mu^{-\kappa-1}| \geq \delta\,(\log x)^{-\kappa} \qquad (\delta > 0,\ 0 < \theta < 1),$$

and therefore $(x \to \infty)$

$$\prod_{\nu=0}^{k}\left(c^\nu + \frac{1}{x}\right)(-1)^k g_\kappa(-x) = O\left(\frac{1}{x^2}\right) + \frac{1}{x}\int_0^1 \frac{dg(t)}{1+xt}$$

$$\geq \frac{\delta}{2}\frac{(\log x)^{-\kappa}}{x} > 0 \qquad \text{for } x \geq x_0.$$

It follows from the next lemma that $(-1)^k g_{k+1}(-x) > 0$ for all large x.

LEMMA 5. *If* $k = 1, 2, \cdots$, *then*

$$g_k(z) = P_{k-1}(z) \prod_{\nu=0}^{k} (c^{-\nu} - z)^{-1},$$

where $P_k(z)$ *is a polynomial of degree* k. *Furthermore,*

$$(-1)^{k+1} c^k z^2 g_k(z) = g_k(1/c^k z) \qquad (z \neq 1, c^{-1}, \cdots, c^{-k}).$$

Proof. We have the formula

$$g_k(z) = \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{(-1)^\nu}{c^{-\nu} - z} = \prod_{\nu=0}^{k} (c^{-\nu} - z)^{-1} \sum_{\mu=0}^{k} \binom{k}{\mu} (-1)^\mu \prod_{\substack{\nu=0\\ \nu\neq\mu}}^{k} (c^{-\nu} - z).$$

But $\prod_{\substack{\nu=0\\ \nu\neq\mu}}^{k} (c^{-\nu} - z) = (-1)^k \left(z^k - z^{k-1}\left(1 + \frac{1}{c} + \cdots + \frac{1}{c^k} - \frac{1}{c^\mu}\right) + \cdots\right)$, and so

$$g_k(z) = \prod_{\nu=0}^{k} (c^{-\nu} - z)^{-1} \left(\left(\frac{1}{c} - 1\right)^k z^{k-1} + \cdots\right).$$

The functional equation for $g_k(z)$ follows from the relations

$$g_k(1/c^k z) = \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{(-1)^\nu}{c^{-\nu} - 1/c^k z} = -z c^k \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{(-1)^{\nu+k} c^{-\nu}}{c^{-\nu} - z}$$

$$= (-1)^{k+1} z c^k \sum_{\nu=0}^{k} \binom{k}{\nu} (-1)^\nu \left(1 + \frac{z}{c^{-\nu} - z}\right) = (-1)^{k+1} z^2 c^k g_k(z).$$

It follows from the functional equation that $g_{2n}(-c^{-n}) = -g_{2n}(-c^{-n})$; that is, $g_{2n}(-c^{-n}) = 0$ $(n = 1, 2, \cdots)$, and if $g_n(z) = 0$, then $g_n(1/c^n z) = 0$.

LEMMA 6. $g_\kappa(-c^{-\kappa/2}) \neq 0$ *for* $\kappa > 0$, $\kappa \neq 2, 4, 6, \cdots$.

Proof. We use Poisson's sum formula in the form

$$\sum_{-\infty}^{+\infty} g(k) = \sum_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-2\pi i k t} g(t)\, dt$$

$\left(\int_{-\infty}^{+\infty} |g(x)|\, dx < \infty,\ g(x) \in V(-\infty, +\infty),\ g(x) = \frac{1}{2}(g(x+0) + g(x-0))\right)$ with

$$g(t) = \begin{cases} (1 - c^t)^\kappa e^{-yt} & (t \geq 0,\ y > 0,\ \kappa > 0), \\ 0 & (t < 0) \end{cases}$$

and obtain the expansion

$$e^{-y} g_\kappa(e^{-y}) = \sum_{0}^{\infty} g(k) = \sum_{-\infty}^{+\infty} \int_0^\infty e^{-t(2\pi i k + y)} (1 - c^t)^\kappa\, dt.$$

But

$$\int_0^\infty e^{-t(2\pi i k+y)} (1 - c^t)^\kappa\, dt = \frac{1}{\log 1/c} \int_0^1 w^{\frac{2\pi i k + y}{\log 1/c} - 1} (1 - w)^\kappa\, dw$$

$$= \frac{1}{\log 1/c} \frac{\Gamma\left(\frac{2\pi i k + y}{\log 1/c}\right) \Gamma(\kappa + 1)}{\Gamma\left(\frac{2\pi i k + y}{\log 1/c} + \kappa + 1\right)},$$

and therefore

$$g_\kappa(e^{-y}) = \frac{e^y \Gamma(\kappa + 1)}{\log 1/c} \sum_{-\infty}^{+\infty} \frac{\Gamma\left(\frac{2\pi i k + y}{\log 1/c}\right)}{\Gamma\left(\frac{2\pi i k + y}{\log 1/c} + \kappa + 1\right)}.$$

This formula obviously furnishes an analytic extension of $g_\kappa(e^{-y})$, if only $\frac{2\pi i k + y}{\log 1/c} \neq -n$ $(n = 0, 1, 2, \cdots)$, that is, $e^{-y} \neq c^{-n}$, and for $y = -(\kappa/2)\log 1/c + \pi i$, the relation $\Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z$ leads to the formula

$$g_\kappa(-c^{-\kappa/2}) = \frac{\pi c^{\kappa/2}\,\Gamma(\kappa+1)}{\log c} \sum_{-\infty}^{+\infty} \frac{1}{\sin \pi\left(i\,\frac{\pi(2k+1)}{\log 1/c} - \frac{\kappa}{2}\right)\left|\Gamma\left(i\,\frac{\pi(2k+1)}{\log 1/c} + 1 + \frac{\kappa}{2}\right)\right|^2}.$$

It follows from the relation

$$\Re\,\frac{1}{\sin \pi\left(i\,\frac{\pi(2k+1)}{\log 1/c} - \frac{\kappa}{2}\right)} = \frac{-\sin\frac{\pi\kappa}{2}\cosh\frac{\pi^2(2k+1)}{\log 1/c}}{\left|\sin \pi\left(i\,\frac{\pi(2k+1)}{\log 1/c} - \frac{\kappa}{2}\right)\right|^2}$$

that

$$g_\kappa(-c^{-\kappa/2})$$

$$= \frac{\pi \sin\frac{\pi\kappa}{2}\, c^{\kappa/2}\,\Gamma(\kappa+1)}{\log 1/c} \sum_{-\infty}^{+\infty} \frac{\cosh\frac{\pi^2(2k+1)}{\log 1/c}}{\left|\sin \pi\left(i\,\frac{\pi(2k+1)}{\log 1/c} - \frac{\kappa}{2}\right)\Gamma\left(i\,\frac{\pi(2k+1)}{\log 1/c} + 1 + \frac{\kappa}{2}\right)\right|^2},$$

which implies that $\sin\frac{\pi\kappa}{2}\, g_\kappa(-c^{-\kappa/2}) > 0$ ($\kappa \neq 2, 4, 6, \cdots$).

We are now in a position to discuss the zeros of $g_\kappa(z)$.

THEOREM 5. *The zeros of* $g_\kappa(z)$ *in* C^* *are all negative and simple. If* $k < \kappa \leq k+1$, *they are exactly* k *in number, and for* $k = 2n$ *and* $k = 2n+1$, *exactly* n *of them lie in the interval* $-c^{-\kappa/2} < x < 0$. *Moreover,* $g_\kappa(-c^{-\kappa/2}) = 0$ *if and only if* $\kappa = 2, 4, 6, \cdots$.

Proof. We already know from Lemma 4 that $g_\kappa(z)$ has at most k zeros in C^*. By induction, we show next that $g_\kappa(z)$ has exactly k negative zeros, and in order to carry out the induction we use the relation $g_{\kappa+1}(z) = g_\kappa(z) - c g_\kappa(cz)$.

If $g_\kappa(z)$ has the zeros $\zeta_1 < \zeta_2 < \cdots < \zeta_k < 0$ with $c\zeta_i < \zeta_{i+1}$ (which is true for $k = 0$), then

$$g_{\kappa+1}(\zeta_i)\, g_{\kappa+1}\left(\frac{\zeta_{i+1}}{c}\right) = -c\, g_\kappa(c\zeta_i)\, g_\kappa\left(\frac{\zeta_{k+1}}{c}\right) < 0 \qquad (i = 1, 2, \cdots, k-1)$$

because of the inequalities $\zeta_i < c\zeta_i$, $\zeta_{i+1}/c < \zeta_{i+1}$. It follows that $g_{\kappa+1}(z)$ has at least one zero between ζ_i and ζ_{i+1}/c ($i = 1, 2, \cdots, k-1$; this statement is empty if $k = 0, 1$); we select one of these zeros and denote it by θ_{i+1}. Since

$$g_{\kappa+1}(\zeta_k) = -c\, g_\kappa(c\zeta_k) < 0 \quad \text{and} \quad g_{\kappa+1}(0) = (1-c)^{\kappa+1} > 0,$$

we see that at least one zero exists between ζ_κ and 0 (this statement is empty if $k = 0$); we select one and denote it by θ_{k+1}. From the relations

$$\operatorname{sgn} g_{\kappa+1}(\zeta_1/c) = \operatorname{sgn} g_\kappa(\zeta_1/c) = (-1)^k$$

and $\operatorname{sgn} g_{\kappa+1}(-x) = (-1)^{k+1}$ (consequence of Lemma 4) for all large x it follows that at least one zero θ_1 of $g_{\kappa+1}$ exists with $\theta_1 < \zeta_1/c$ (if $k = 0$, then $g_{\kappa+1}(0) > 0$, $g_{\kappa+1}(-x) < 0$ when x is large, and $g_{\kappa+1}(z)$ has at least one zero $\theta_1 < 0$). It follows that $g_{\kappa+1}(z)$ has exactly $k+1$ zeros, and this is true for all k if we show that $c\theta_i < \theta_{i+1}$ $(i = 1, 2, \cdots, k)$. But this is true because $\theta_1 < \zeta_1/c$, $\zeta_1 < \theta_2$ (that is, $c\theta_1 < \theta_2$), $\theta_{i+1} < \zeta_{i+1}/c$, $\zeta_{i+1} < \theta_{i+2}$ for $i = 1, 2, \cdots, k-1$ (that is, $c\theta_j < \theta_{j+1}$ for $j = 2, 3, \cdots, k$). Let C_κ be the curve consisting of

$$z = it \quad (-c^{-\kappa/2} \leq t \leq c^{-\kappa/2}) \quad \text{and} \quad z = c^{-\kappa/2} e^{i\phi} \quad \left(\frac{\pi}{2} \leq \phi \leq \frac{3}{2}\pi\right);$$

then the quantity

$$I_\kappa = \int_{C_\kappa} \frac{g'_\kappa(z)}{g_\kappa(z)}\, dz = c^{-\kappa/2} \int_{C_0} \frac{g'_\kappa(wc^{-\kappa/2})}{g_\kappa(wc^{-\kappa/2})}\, dw$$

is constant for every interval $2n < \kappa < 2n+2$ (because of Lemma 6 and the fact that all zeros of $g_\kappa(z)$ are negative). But we know from Lemma 5 that $I_{2n+1} = 2\pi i n$, and this implies that $g_\kappa(z)$ $(2n < \kappa < 2n+2)$ has exactly n zeros in $-c^{-\kappa/2} < x < 0$. The proof of Theorem 5 is now complete if we observe that $g_{2n}(z)$ has exactly $n-1$ zeros in the interval $-c^{-n} < x < 0$ $(n = 1, 2, \cdots)$, which also follows from Lemma 5.

For $c = 1/2$, a calculation carried out by R. Yeaman and F. Lether (University of Utah Computer Center) showed that $g_\kappa(z) \neq 0$ $(-1 \leq z \leq 0)$ for $\kappa < \kappa_0 = 2.6058$ $\left(\text{error less than } \frac{1}{2}10^{-4}\right)$, and $g_{\kappa_0}(-1) = 0$. The foregoing proof, as well as some more calculations, indicate that the zeros of $g_\kappa(z)$ increase with κ. For $f_\kappa(z)$, the situation seems to be similar.

5. HURWITZ'S THEOREM ON THE ZEROS OF BESSEL FUNCTIONS

As an application of Theorem 3, we give a new proof of Hurwitz's theorem on the zeros of Bessel functions of negative order (for earlier proofs, see Hille and Szegö [7] and the literature quoted there). We begin with some notation and known relations.

We write

$$\phi_\nu(z) = \frac{J_\nu(2\sqrt{z})}{z^{\nu/2}} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\,\Gamma(n+\nu+1)} \quad (\nu \neq -1, -2, \cdots).$$

The (entire and real) function $\phi_\nu(z)$ has order $1/2$ and satisfies the relations

$$\phi'_\nu(z) = -\phi_{\nu+1}(z), \tag{17}$$

$$\phi_\nu(z) = (\nu+1)\phi_{\nu+1}(z) + z\phi'_{\nu+1}(z), \tag{18}$$

$$z\phi''_\nu(z) + (\nu+1)\phi'_\nu(z) + \phi_\nu(z) = 0. \tag{19}$$

It follows from (19) (and the definition of $\phi_\nu(z)$) that $\phi_\nu(z)$ has simple zeros only.

LEMMA 7. *If* $k = 1, 2, \cdots$ *and* $-k < \nu < -k+1$, *then* $\phi_\nu(z)$ *has exactly* $k-1$ *zeros that are not positive.*

Proof. We proceed by induction, using Theorem 3 and (18), and starting with the well-known case $k = 1$ (Lommel's Theorem; see for example Watson [32, p. 482]).

In order to apply Theorem 3, we must prove that $\phi_\nu(z)$ has infinitely many positive zeros (by (18), this follows immediately from the case $k = 0$), and that the zeros of $\phi_\nu(z)$ and $\phi_{\nu+1}(z)$ interlace in the sense of Theorem 3. By (18), there is at least one zero of $\phi_\nu(z)$ between two zeros of $\phi_{\nu+1}(z)$, and by (17), there is at most one. It remains to show that there is no zero of $\phi_\nu(z)$ between 0 and a, where a denotes the smallest positive zero of $\phi_{\nu+1}(z)$. But $\phi_\nu(0)\phi_{\nu+1}(0) < 0$ $(\nu < -1)$, and it follows from (17) that $\phi_\nu(z) \neq 0$ for $0 \leq z \leq a$.

Remark. Most of the arguments of the foregoing proof can be found in Watson [32, pp. 479-480]. The assumption $\nu < -1$ was used only at the end of the foregoing proof. If $\nu > -1$, then $\phi_\nu(z)$ has one zero in the interval (0, a), which accounts for the fact that $J_\nu(z)$ has no complex zero.

The following lemma gives more precise information on the location of the zeros of $\phi_\nu(z)$.

LEMMA 8. *Let* $-k < \nu < -k+1$. *The function* $\phi_\nu(z)$ *has exactly one negative zero if* $k = 2n$ $(n = 1, 2, \cdots)$, *no negative zero if* $k = 2n - 1$.

Proof. For $k = 1, 2$, this follows from Lemma 7 (if $k = 2$, then $\phi_\nu(z)$ has one zero, which is not positive). We proceed by induction and observe first (from the power series representation of $\phi_\nu(z)$) that $\phi_\nu(-x) \to \infty$ as $x \to \infty$.

(a) Assume that $k = 2n + 1$ $(n = 1, 2, \cdots)$. In this case, $\phi_\nu(0) > 0$, which implies that $\phi_\nu(z)$ has an even number of negative zeros. It follows from (17) that $\phi_\nu(z)$ has at most two negative zeros (otherwise $\phi_{\nu+1}(z)$ would have more than one negative zero). Assume that $\phi_\nu(\alpha) = \phi_\nu(\beta) = 0$ $(\alpha < \beta < 0)$; then $\phi_{\nu+1}(\gamma) = 0$ for some γ $(\alpha < \gamma < \beta)$, $\phi_{\nu+1}(\alpha) > 0$, $\phi_{\nu+1}(\beta) < 0$ (by (17)). It follows from (17) and (18) that $0 = \phi_\nu(\beta) = (\nu + 1)\phi_{\nu+1}(\beta) - \beta\phi_{\nu+2}(\beta) > 0$, which contradicts our assumption.

(b) Assume that $k = 2n$ $(n = 2, 3, \cdots)$. In this case, $\phi_\nu(0) < 0$, and $\phi_\nu(z)$ has at least one negative zero (because $\phi_\nu(-x) \to \infty$ as $x \to \infty$). It also has at most one negative zero, since $\phi'_\nu(x) = -\phi_{\nu+1}(x) < 0$. From the definition of $\phi_\nu(z)$ and Lemmas 7 and 8 we immediately deduce the theorem of Hurwitz:

If $-2k < \nu < -2k + 1$ $(k = 1, 2, \cdots)$, *then* $J_\nu(z)$ *has (besides its real zeros)* $4k - 2$ *complex zeros, two of which lie on the imaginary axis.*

If $-2k + 1 < \nu < -2k + 2$ $(k = 1, 2, \cdots)$, *then* $J_\nu(z)$ *has (besides its real zeros)* $4k - 4$ *complex zeros, and no zero lies on the imaginary axis.*

REFERENCES

1. J. Bendat and S. Sherman, *Monotone and convex operator functions*, Trans. Amer. Math. Soc. 79 (1955), 58-71.

2. E. Borel, *Leçons sur les fonctions entières*, Second Edition, Gauthier-Villars, Paris, 1921.

3. S. Chapman, *On non-integral orders of summability of series and integrals*, Proc. London Math. Soc. (2) 9 (1911), 369-409.

4. A. C. Climescu, *Sur la classe des fonctions analytiques qui gardent les demi-plans déterminés par l'axe réel*, Ann. Sci. Univ. Jassy Sect. I. 28 (1942), 31-138.

5. J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de Taylor*, J. Math. Pures Appl. (4) 8 (1892), 101-186.

6. F. Hausdorff, *Summationsmethoden und Momentfolgen, I*, Math. Z. 9 (1921), 74-109.

7. E. Hille and G. Szegö, *On the complex zeros of the Bessel functions*, Bull. Amer. Math. Soc. 49 (1943), 605-610.

8. W. Kaplan, *Zeros of analytic functions and the moment problem*, Ann. Acad. Sci. Fenn. Ser. A.I. 250/17 (1958), 11 pp.

9. A. Korányi, *Note on the theory of monotone operator functions*, Acta Sci. Math. Szeged 16 (1955), 241-245.

10. ———, *On a theorem of Löwner and its connections with resolvents of selfadjoint transformations*, Acta Sci. Math. Szeged 17 (1956), 63-70.

11. B. Kuttner, *On discontinuous Riesz means of type* n, J. London Math. Soc. 37 (1962), 354-364.

12. ———, *The high indices theorem for discontinuous Riesz means*, J. London Math. Soc. 39 (1964), 635-648.

13. D. F. Lawden, *The function* $\sum_{n=1}^{\infty} n^r z^n$ *and associated polynomials*, Proc. Cambridge Philos. Soc. 47 (1951), 309-314.

14. E. LeRoy, *Sur les séries divergentes et les fonctions définies par un développement de Taylor*, Ann. Fac. Sci. Univ. Toulouse (2) 2 (1900), 317-430.

15. V. I. Levin, *Concerning a problem of S. Ramanujan*, Uspehi Mat. Nauk (N.S.) 5 no. 3 (37) (1950), 161-166 (Russian).

16. M. Marden, *On the zeros of rational functions having prescribed poles, with applications to the derivative of an entire function of finite genre*, Trans. Amer. Math. Soc. 66 (1949), 407-418.

17. E. P. Merkes, *On typically-real functions in a cut plane*, Proc. Amer. Math. Soc. 10 (1959), 863-868.

18. W. Miesner, *On the convergence fields of Nörlund means*, Proc. London Math. Soc. (3) 15 (1965), 495-507.

19. W. Miesner and E. Wirsing, *On the zeros of* $\sum (n+1)^{\kappa} z^n$, J. London Math. Soc. (to appear).

20. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin, 1924.

21. A. Peyerimhoff, *On convergence fields of Nörlund means*, Proc. Amer. Math. Soc. 7 (1956), 335-347.

22. ———, *On the modulus of power series of a certain type*, J. London Math. Soc. 40 (1965), 260-261.

23. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Second Edition, Springer, Berlin, 1954.

24. M. Riesz, *Sur l'équivalence de certaines méthodes de sommation*, Proc. London Math. Soc. (2) 22 (1924), 412-419.

25. H. F. Sandham, *A logarithmic transcendent*, J. London Math. Soc. 24 (1949), 83-91.

26. R. Seall and M. Wetzel, *Some connections between continued fractions and convex sets*, Pacific J. Math. 9 (1959), 861-873.

27. J. S. Thale, *Univalence of continued fractions and Stieltjes transforms*, Proc. Amer. Math. Soc. 7 (1956), 232-244.

28. C. Truesdell, *On a function which occurs in the theory of the structure of polymers*, Ann. of Math. (2) 46 (1945), 144-157.

29. M. L. Veržbinskiĭ, *On the distribution of the roots of the* L-*transforms of entire transcendental functions*, Mat. Sb. N.S. 22 (64) (1948), 391-424 (Russian).

30. H. S. Wall, *Continued fractions and totally monotone sequences*, Trans. Amer. Math. Soc. 48 (1940), 165-184.

31. ———, *A class of functions bounded in the unit circle*, Duke Math. J. 7 (1940), 146-153.

32. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, 1944.

33. D. Zeitlin, *Two methods for the evaluation of* $\sum_{k=0}^{\infty} k^n x^k$, Amer. Math. Monthly 68 (1961), 986-989.

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