NAǏMARK'S MOMENT THEOREM

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There are intimate connections between the Naǐmark-Nagy dilation theory, the theory of group representations, and the theory of integration with respect to positive operator-valued measures. The object of this largely expository article is to inspect some of the details, raise a few questions, and record some partial answers. The principal results are some minor improvements on Naǐmark's dilation theorem and Naǐmark's moment theorem; these results are then related to the Nagy dilation of a contraction, and the article concludes with a historical note.

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1. COMPLEMENTS TO NAǏMARK'S DILATION THEOREM

Our basic references for Hilbert space and measure theory are the books [9], [10] of P. R. Halmos; thus *inner products* are denoted by (x, y), continuous linear mappings are called *operators*, a nonempty class of sets is called a *ring* if it contains differences and finite unions, and a nonnegative extended real valued function defined on a ring of sets is called a *measure* if it is countably additive and vanishes on the empty set. Before stating Naǐmark's dilation theorem, we review some concepts that are not quite standard. The first definition is motivated by the finite additivity of the classical Jordan content:

Definition 1. An *operator content* is a quadruple $(X, \mathscr{R}, \mathscr{H}, F)$, where X is a set, $\mathscr{R}$ is a ring of subsets of X, $\mathscr{H}$ is a Hilbert space, and F is a function defined on $\mathscr{R}$ whose values are positive operators in $\mathscr{H}$, such that $F(M \cup N) = F(M) + F(N)$ whenever M and N are disjoint sets in $\mathscr{R}$.

To simplify notation, let us fix an operator content $(X, \mathscr{R}, \mathscr{H}, F)$; we may then refer briefly to F or to $(\mathscr{H}, F)$ as an operator content. If M and N are sets in $\mathscr{R}$ such that $M \subset N$, then $F(M) \leq F(N)$; that is, F is *monotone*. It follows that if M_n is an increasing sequence of sets in $\mathscr{R}$ whose union M is also in $\mathscr{R}$, then $F(M_n)$ is an increasing sequence of (positive) Hermitian operators bounded above by F(M), and therefore $\text{LUB } F(M_n) \leq F(M)$ [2, p. 6, Proposition 1].

Definition 2. We say that F is *continuous* if $F(M) = \text{LUB } F(M_n)$ whenever M_n is an increasing sequence in $\mathscr{R}$ whose union M is also in $\mathscr{R}$; briefly, $M_n \uparrow M$ implies $F(M_n) \uparrow F(M)$. We then call F an *operator measure*.

Continuity is equivalent to the condition that for each vector x in $\mathscr{H}$, the set function $M \to (F(M)x, x)$ is a (finite) measure on $\mathscr{R}$ [2, p. 8, Theorem 1]. In order that F be continuous, it is necessary and sufficient that F be countably additive in the weak (or the strong) operator topology (see [10, p. 59, Theorem 3] and [2, Section 3]). If $\mathscr{H}$ is one-dimensional, the values of F may be regarded as real numbers, thus operator measures are a generalization of finite (numerical) measures.

Definition 3. We say that F is *bounded* if there exists a positive real number k such that $\|F(M)\| \leq k$ for all M in $\mathscr{R}$ (equivalently, $F(M) \leq kI$ for all M in $\mathscr{R}$, where I is the identity operator in $\mathscr{H}$).

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An operator measure defined on a σ-ring is necessarily bounded [2, p. 13, Theorem 5]. An operator content whose values are projections (that is, Hermitian idempotents) is bounded. If $X \in \mathscr{R}$, then F is bounded because $0 \leq F(M) \leq F(X)$ for all M in $\mathscr{R}$.

Definition 4. If $X \in \mathscr{R}$ and $F(X) = I$, we say that F is *normalized.*

Definition 5. If F is normalized and projection-valued, we say that F is *spectral.* An operator measure that is spectral is called a *spectral measure.*

In general we assume merely that F is an operator content; the properties of continuity, boundedness, normalization, and spectrality will be stated explicitly when they are called for.

Definition 6. An operator content $(X, \mathscr{R}, \mathscr{D}, E)$ is called a *dilation* of the operator content $(X, \mathscr{R}, \mathscr{H}, F)$ if $\mathscr{H}$ is a closed linear subspace of $\mathscr{D}$, and if moreover

$$F(M)x = PE(M)x$$

for all x in $\mathscr{H}$ and M in $\mathscr{R}$, where P is the projection of $\mathscr{D}$ on $\mathscr{H}$. We say briefly that $(\mathscr{D}, E)$ is a dilation of $(\mathscr{H}, F)$, or that $(\mathscr{H}, F)$ is a *compression* of $(\mathscr{D}, E)$ [8, p. 126]. If, moreover, the set of vectors $F(M)x$ $(x \in \mathscr{H},\ M \in \mathscr{R})$ is total in $\mathscr{D}$, we say that $(\mathscr{D}, E)$ is a *minimal* dilation of $(\mathscr{H}, F)$.

Definition 7. Two dilations $(\mathscr{D}_1, E_1)$ and $(\mathscr{D}_2, E_2)$ of $(\mathscr{H}, F)$ are said to be *isomorphic over* $\mathscr{H}$ if there exists an isometric linear mapping W of $\mathscr{D}_1$ onto $\mathscr{D}_2$ such that $WE_1(M) = E_2(M)W$ for all M in $\mathscr{R}$, and $Wx = x$ for all x in $\mathscr{H}$.

We can now state Naĭmark's dilation theorem concisely as follows [14]:

NAĬMARK'S DILATION THEOREM. *Let* $(X, \mathscr{R}, \mathscr{H}, F)$ *be a normalized operator content.*

Existence: *There exists a minimal spectral dilation* $(\mathscr{D}, E)$ *of* $(\mathscr{H}, F)$.

Uniqueness: *Any two minimal spectral dilations of* $(\mathscr{H}, F)$ *are isomorphic over* $\mathscr{H}$.

Continuity: *If* F *is continuous, then so is* E.

Naĭmark's proof makes crucial use of the assumption that F is normalized (see [14] and [24]) and therefore bounded; at the expense of uniqueness, the theorem can be extended to bounded operator contents:

THEOREM 1. *Let* $\mathscr{R}$ *be a ring of subsets of a set* X, *and suppose* F *is a finitely additive set function on* $\mathscr{R}$ *whose values are positive operators in a Hilbert space* $\mathscr{H}$. *Assume that there exists a positive real number* k *such that* $\|F(M)\| \leq k$ *for all* M *in* $\mathscr{R}$.

Then there exists a Hilbert space $\mathscr{D}$ *containing* $\mathscr{H}$ *as closed linear subspace, and a finitely additive set function* E *on* $\mathscr{R}$ *whose values are projection operators in* $\mathscr{D}$, *such that*

$$F(M)x = kPE(M)x$$

for all x *in* $\mathscr{H}$ *and* M *in* $\mathscr{R}$, *where* P *is the projection of* $\mathscr{D}$ *on* $\mathscr{H}$. *If* F *is continuous, we can take* E *to be continuous also.*

Our hypothesis, briefly, is that the operator content F is bounded by k; the conclusion is that $k^{-1}F$ possesses a projection-valued dilation E, and that E can be taken to be continuous if F is continuous. We remark that k is *any* upper bound for the $\|F(M)\|$; it is not assumed to be minimal.

Proof of Theorem 1. We can suppose that $k = 1$ (replace F by $k^{-1}F$); then $0 \leq F(M) \leq I$ for all M in $\mathscr{R}$. (If T is any operator such that $0 \leq T \leq I$, it is easy to produce a projection that dilates T [8, p. 128, Theorem 2]; however, our problem is to dilate a family of such operators simultaneously and in a coherent manner.) Let $\mathscr{A}$ be the class of all sets $A \subset X$ such that $A \cap M \in \mathscr{R}$ for every M in $\mathscr{R}$; then $\mathscr{A}$ is an algebra containing $\mathscr{R}$ as an ideal, and the formula

$$G(A) = \text{LUB}\{F(M): M \subset A, \ M \in \mathscr{R}\}$$

defines a finitely additive extension of F to $\mathscr{A}$ such that $0 \leq G(A) \leq I$ for all A in $\mathscr{A}$. Moreover, if F is continuous, then so is G [2, p. 17, Theorem 8].

Changing notation, we can suppose that $\mathscr{R}$ is already an algebra and $F(X) \leq I$. Let Y be the result of adjoining to X a new point ω, and let $\mathscr{S}$ be the algebra consisting of the sets M and $M \cup \{\omega\}$, where M varies over $\mathscr{R}$. Extend F to $\mathscr{S}$ by the formula

$$F(M \cup \{\omega\}) = F(M) + (I - F(X)).$$

(This ingenious idea, the key to the whole proof, is due to J. G. Stampfli; it was communicated to me by R. G. Douglas.) The extra feature of the extension is that it is normalized ($F(Y) = I$); it therefore possesses a Naĭmark minimal spectral dilation, whose restriction to $\mathscr{R}$ is the promised set function E.

There is a useful application of Theorem 1 to the theory of integration with respect to operator measures:

COROLLARY. *If* F *is an operator measure defined on a* σ*-ring* $\mathscr{R}$, *and if* k *is the least upper bound of* $\|F(M)\|$ *as* M *varies over* $\mathscr{R}$, *then*

$$\left\|\int f\,dE\right\| \leq k\,\|f\|_\infty$$

for every complex-valued bounded measurable function f.

Proof. As noted earlier, k is necessarily finite. With notation as in Theorem 1, we have the relation

$$F(M)x = kPE(M)x$$

for all x in $\mathscr{H}$ and M in $\mathscr{R}$, and therefore

$$\left(\int f\,dF\right)x = kP\left(\int f\,dE\right)x$$

for all x in $\mathscr{H}$ (see [2, p. 27, Definition 9]). Then, for all x in $\mathscr{H}$, we have the inequalities

$$\left\|\left(\int f\,dF\right)x\right\| \leq k\left\|\left(\int f\,dE\right)x\right\| \leq k\,\|f\|_\infty\,\|x\|,$$

by an elementary property of projection-valued operator measures [2, p. 35, Theorem 16].

The proof of the corollary is pleasantly brief, but the proof of Naĭmark's dilation theorem is not; from the point of view of integration theory, it would be nice to have a straightforward proof of the corollary that does not resort to dilation theory.

2. POSITIVE DEFINITE vs. POSITIVE TYPE

Let G be a group (with multiplicative notation, and with neutral element e). A complex-valued function p on G is said to be *positive definite* if

$$\sum_{s,t} p(t^{-1} s) c_s \bar{c}_t \geq 0$$

for every finitely nonzero family (c_s) of complex numbers indexed by G. The classical result, for the case that G is the group of integers, is due to G. Herglotz [12]: every such function (really a bilateral sequence) may be represented by means of a suitable measure on the circle group (see [18, p. 116] or [16, p. 411]). More generally, if G is any locally compact abelian group and X is the character group of G [13, p. 134], then every continuous positive definite function on G may be represented by means of a suitable measure on X; this was proved for the group of real numbers by S. Bochner [3, p. 76, Theorem 23], and for the general case by A. Weil [25, p. 122] and D. A. Raĭkov [17]. There is even a theorem of this type that holds for arbitrary topological groups (see [16, p. 393, Theorem 1]).

This is, so to speak, the one-dimensional case. How should the concept of positive definiteness be extended to functions whose values are operators in a not necessarily one-dimensional Hilbert space $\mathscr{H}$? The literature indicates two answers:

Definition 8. A family (T_s) of operators in $\mathscr{H}$, indexed by a group G, is said to be of *positive type* if

$$\sum_{s,t} (T_{t^{-1}s} x_s, x_t) \geq 0$$

for every finitely nonzero family (x_s) of vectors in $\mathscr{H}$ indexed by G.

Definition 9. A family (T_s) of operators in $\mathscr{H}$, indexed by a group G, will be called *positive definite* if for each vector x in $\mathscr{H}$, the complex-valued function $s \to (T_s x, x)$ is positive definite in the classical sense, that is,

$$\sum_{s,t} (T_{t^{-1}s} x, x) c_s \bar{c}_t \geq 0$$

for every finitely nonzero family (c_s) of complex numbers indexed by G.

The terminology in the literature varies. The concept in Definition 8 appears in the work of Naĭmark [15], who calls it "positive definite"; Nagy calls it "type positif" [23], and A. Devinatz calls it "strongly positive definite" [6]. The concept in Definition 9 is mentioned by J. Bram [4, p. 79, part (b) of Theorem B], and it is implicit in Nagy's first paper on the dilation of contractions [22]. Definition 8 has proved to be the better suited for generalization [24], as we shall note in the next section.

It is obvious that every family of positive type is positive definite; I do not know if the converse holds in general, but it does when the group is abelian:

THEOREM 2. *Every positive definite family of operators indexed by an abelian group is of positive type.*

Proof. Denote the group by G, the Hilbert space by $\mathscr{H}$, and the family of operators by (T_s). Regard G as a discrete topological group, and let X be the character group of G, that is, the group of all homomorphisms α of G into the circle group $K = \{\lambda: |\lambda| = 1\}$ (see [13, p. 137]). With the topology of pointwise convergence, X is compact [13, p. 153]. If $s \in G$, we write $\hat{s}$ for the function on X defined by $\hat{s}(\alpha) = \alpha(s)$; thus $\hat{s}$ is a continuous character of X.

To each vector x in $\mathscr{H}$ there corresponds, by the Herglotz-Bochner-Weil-Raĭkov theorem, a unique Baire measure μ_x on X such that

$$(1) \qquad (T_s x, x) = \int \hat{s}\, d\mu_x$$

for all s in G (see [16, p. 410]).

Let $\mathscr{A}$ be the linear span of the functions $\hat{s}$, that is, let $\mathscr{A}$ be the class of all functions f on X such that

$$(2) \qquad f = \sum_{i=1}^{n} c_i \hat{s}_i$$

for suitable complex numbers $c_1, \cdots, c_n$ and elements $s_1, \cdots, s_n$ of G (such functions are sometimes called trigonometric polynomials). Since

$$(st)^\wedge = \hat{s}\hat{t}, \qquad (s^{-1})^\wedge = (\hat{s})^-, \qquad \text{and } \hat{e} = 1,$$

it follows that $\mathscr{A}$ is a complex *-algebra containing the constant functions; since $\mathscr{A}$ obviously separates the points of X, it follows from the Weierstrass-Stone theorem that $\mathscr{A}$ is uniformly dense in the algebra $\mathscr{C}(X)$ of all continuous complex-valued functions on X.

We now establish an operator-valued correspondence $f \to T(f)$ $(f \in \mathscr{A})$ as follows. If $f \in \mathscr{A}$ is written as in (2), we propose to define

$$(3) \qquad T(f) = \sum_{i=1}^{n} c_i T_{s_i};$$

since it is clear from (1) and (2) that the relation

$$\left(\left[\sum_{i=1}^{n} c_i T_{s_i}\right] x, x\right) = \int f\, d\mu_x$$

holds for all x in $\mathscr{H}$, the operator on the right side of (3) depends only on f and not on the particular representation (2). Thus T(f) is well-defined by (3), and

$$(4) \qquad (T(f)x, x) = \int f\, d\mu_x$$

for all f in $\mathscr{A}$ and x in $\mathscr{H}$.

The correspondence $f \to T(f)$ is obviously linear, and it is immediate from (4) that $T(f) \geq 0$ whenever $f \geq 0$. Also, $T(\hat{s}) = T_s$ for all s in G, and in particular $T(1) = T_e$. It follows easily that

$$\|T(f)\| \leq 2\,\|f\|_\infty\,\|T_e\|$$

for all f in $\mathscr{A}$ (where $\|f\|_\infty$ is the least upper bound of $|f(\alpha)|$ as α varies over X). Thus the correspondence $f \to T(f)$ ($f \in \mathscr{A}$) is continuous in norm; it can therefore be extended by continuity to a correspondence $f \to T(f)$ ($f \in \mathscr{C}(X)$), and it is easy to see that the extension retains the property that $T(f) \geq 0$ whenever $f \geq 0$ (see [2, p. 78]).

By the operatorial Riesz-Markoff theorem proved in [2, p. 39], there exists a unique operator measure F on the Baire sets of X such that

$$T(f) = \int f\,dF$$

for all f in $\mathscr{C}(X)$; in particular $T(\hat{s}) = \int \hat{s}\,dF$, that is,

$$T_s = \int \hat{s}\,dF \tag{5}$$

for all s in G.

We summarize this result in the form of a *generalized Herglotz theorem:* if G is a discrete abelian group with character group X, and if (T_s) is a family of operators indexed by G, then (T_s) is positive definite if and only if there exists an operator measure F on the Baire sets of X such that (5) holds. (The "if" part is trivial; see [18, p. 117].) Moreover, since the trigonometric polynomials are uniformly dense in $\mathscr{C}(X)$, the operator measure F is uniquely determined by (5) (see [2, p. 27, Theorem 10, and p. 29, Theorem 11]).

Let $k = \|F(X)\|$. By Theorem 1, there exists a projection-valued dilation $(\mathscr{D}, E)$ of $(\mathscr{H}, k^{-1} F)$; thus

$$F(M)x = kPE(M)x$$

for all x in $\mathscr{H}$ and all Baire sets M of X, where P is the projection of $\mathscr{D}$ on $\mathscr{H}$. It follows easily that

$$\left(\int f\,dF\right)x = kP\left(\int f\,dE\right)x \tag{6}$$

for all x in $\mathscr{H}$ and all complex-valued bounded Baire functions f on X [2, p. 27]. For each s in G define

$$V_s = \int \hat{s}\,dE. \tag{7}$$

It is easy to see that $V_{st} = V_s V_t$ for all s and t, $V_t = V_s^*$ when $t = s^{-1}$, and V_e is a projection (see [2, p. 27, Theorem 10, and p. 34, Theorem 15]); so to speak, $s \to V_s$ is a "partially unitary" representation of G in $\mathscr{D}$. From (5) to (7) we see that

$$T_s x = \left(\int \hat{s}\, dF\right) x = kP\left(\int \hat{s}\, dE\right) x = kPV_s x$$

for all x in $\mathscr{H}$ and s in G; it follows that if (x_s) is any finitely nonzero family of vectors in $\mathscr{H}$ indexed by G, then

$$(T_{t^{-1}s} x_s, x_t) = k(PV_t^* V_s x_s, x_t) = k(V_s x_s, V_t x_t),$$

and therefore

$$\sum_{s,t} (T_{t^{-1}s} x_s, x_t) = k \left\| \sum_s V_s x_s \right\|^2 \geq 0.$$

This shows that the family (T_s) is of positive type, and it concludes the proof of Theorem 2. Note also that

$$k = \|F(X)\| = \left\| \int 1\, dF \right\| = \left\| \int \hat{e}\, dF \right\| = \|T_e\|;$$

citing (5) and the corollary of Theorem 1, we conclude that

$$\|T_s\| \leq k \|\hat{s}\|_\infty = k = \|T_e\|.$$

More generally:

COROLLARY. *If* (T_s) *is a positive definite family of operators indexed by an arbitrary group* G, *then*

$$\|T_s\| \leq \|T_e\| \tag{8}$$

for all s *in* G.

Proof. Let s be any given element of G, and let H be the subgroup of G generated by s. Then H is abelian, and the family $h \to T_h$ $(h \in H)$ is also positive definite; therefore $\|T_s\| \leq \|T_e\|$, by the preceding argument. The related inequality

$$|(T_s x, x)| \leq (T_e x, x), \tag{9}$$

valid for every s in G and every vector x, is elementary [16, p. 391].

Theorem 2 and its corollary make no mention of dilation theory in their statements, but I do not know how to prove them without using dilation theory.

3. COMPLEMENTS TO NAĬMARK'S MOMENT THEOREM

If (T_s) is a family of operators indexed by a topological group G, there are obvious definitions of weak and strong continuity for the family; strong continuity implies weak continuity, and the two notions coincide when the T_s are unitary [24, p. 22].

We return to the theme, touched in Section 2, of representing functions on a locally compact abelian group by means of measures on its character group. For operator-valued functions, the definitive result is due to Naĭmark [15] (for the definition of a regular weakly Borel operator measure, see [2, p. 46]):

NAĬMARK'S MOMENT THEOREM. *Let* G *be a locally compact abelian group, with character group* X, *and let* (T_s) *be a weakly continuous family of operators in a Hilbert space* $\mathscr{H}$, *indexed by* G, *such that* $T_e = I$. *The following conditions on the family are equivalent:*

(a) (T_s) *is of positive type.*

(b) (T_s) *is positive definite.*

(c) *There exists a regular weakly Borel operator measure* F *(necessarily unique) on* X *such that*

$$T_s = \int \hat{s}\, dF \tag{10}$$

for all s *in* G.

This theorem is cited by Bram [4, p. 79] as being contained in Naĭmark's paper [15]; strictly speaking, Naĭmark proves the equivalence of (a) and (c), but (a) and (b) are equivalent by Theorem 2. The hypothesis $T_e = I$ shows up in part (c) as $F(X) = I$; this hypothesis can be eliminated:

THEOREM 3. *Naĭmark's moment theorem remains true if the hypothesis* $T_e = I$ *is omitted.*

Proof. (a) and (b) are equivalent by Theorem 2.

(c) implies (b): Fix a vector x in $\mathscr{H}$, and let μ be the regular weakly Borel measure on X defined by

$$\mu(M) = (F(M)x, x)$$

(see [2, p. 44, proof of Theorem 20]). Then

$$(T_s x, x) = \int \hat{s}\, d\mu$$

for all s in G [2, p. 27], and the function $s \to (T_s x, x)$ is positive definite by an elementary argument [18, p. 117].

(b) implies (c): The argument for this is given in [1]; for the sake of completeness, we sketch it here. For each vector x in $\mathscr{H}$, the function $s \to (T_s x, x)$ is continuous and positive definite; by the Herglotz-Bochner-Weil-Raĭkov theorem, there exists a unique finite regular weakly Borel measure μ_x on X such that

$$(T_s x, x) = \int \hat{s}\, d\mu_x \tag{11}$$

for all s in G. For each pair of vectors x, y in $\mathscr{H}$, the formula

$$\mu_{x,y} = \frac{1}{4}\{\mu_{x+y} - \mu_{x-y} + i\mu_{x+iy} - i\mu_{x-iy}\}$$

defines a regular weakly Borel complex measure (briefly, complex measure) such that

$$(T_s x, y) = \int \hat{s}\, d\mu_{x,y} \tag{12}$$

for all s in G. If μ and ν are complex measures on X such that

$$\int \hat{s}\, d\mu = \int \hat{s}\, d\nu$$

for all s in G, then $\mu = \nu$ by the uniqueness theorem for Fourier-Stieltjes transforms (see [5, p. 88, Theorem 1] and [25, p. 122]); from this, and (12), it follows that $\mu_{x,y}$ is a sesquilinear function of x and y. Moreover, for any weakly Borel set M in X we see that

$$|\mu_{x,x}(M)| = \mu_x(M) \leq \mu_x(X) = \int \hat{e}\, d\mu_x = (T_e x, x) \leq \|T_e\| \, \|x\|^2 ;$$

it follows that for fixed M, $\mu_{x,y}(M)$ is a bounded sesquilinear form in x and y [10, p. 33], thus there is a unique operator F(M) such that

$$(F(M)x, y) = \mu_{x,y}(M)$$

for all x and y [10, p. 39]. In particular,

$$(F(M)x, x) = \mu_x(M) \tag{13}$$

for all M and x; it follows at once that F is an operator measure on the weakly Borel sets of X [2, p. 8, Theorem 1], and that it is regular (see [2, p. 44, proof of Theorem 20]). From (11) and (13) we see that

$$(T_s x, x) = \int \hat{s}\, d\mu_x = \left(\left[\int \hat{s}\, dF\right] x, x\right),$$

and the formula (10) is established.

The assertion concerning the uniqueness of F follows from the fact, cited earlier in the proof, that two complex measures on X are identical if they assign the same integral to the functions $\hat{s}$. (We remark that Theorem 3 includes the generalized Herglotz theorem mentioned in the proof of Theorem 2, but the proof of Theorem 3 lies deeper.)

In Naǐmark's paper [15] it is also shown that condition (a) of the moment theorem (assuming weak continuity and $T_e = I$) is equivalent to the following condition:

(d) *There exists a Hilbert space* $\mathscr{D}$ *containing* $\mathscr{H}$ *as closed linear subspace, and a strongly continuous unitary representation* $s \to U_s$ *of* G *in* $\mathscr{D}$ *such that*

$$T_s x = P U_s x$$

for all x *in* $\mathscr{H}$ *and* s *in* G, *where* P *is the projection of* $\mathscr{D}$ *on* $\mathscr{H}$.

The nontrivial part of the equivalence is to prove that (a) implies (d). One method is to take the normalized operator measure F given by part (c), to let $(\mathscr{D}, E)$ be the Naǐmark minimal spectral dilation of $(\mathscr{H}, F)$, and to define $U_s = \int \hat{s}\, dE$ for all s in G.

There is a simpler procedure; the statements (a) and (d) do not involve X, and Nagy showed that their equivalence can be proved directly, without appeal to character theory or measure theory [23]. Indeed, Nagy's proof works for arbitrary topological groups (not necessarily abelian or locally compact). The definitive result in this essentially algebraic direction was also obtained by Nagy [24, Principal Theorem]; the setting for this beautiful theorem is an arbitrary involutive semigroup.

In view of condition (d), the Naĭmark moment theorem yields the following striking result:

If (T_s) *is a family of operators indexed by a locally compact abelian group, and if* $T_e = I$, *then the family is weakly continuous and positive definite if and only if it is strongly continuous and of positive type.*

I do not know whether the preceding statement is true without the normalization assumption $T_e = I$. A related problem is to prove the statement without making use of character theory or dilation theory.

4. THE NAGY UNITARY DILATION OF A CONTRACTION

The proof of Theorem 2 is much simpler when G is the group of integers. Then X is the circle group K, and the classical versions of the Herglotz and Weierstrass theorems are adequate for the proof. We summarize by restating Theorem 3 for the group of integers:

THEOREM 4. *If* (T_n) *is a bilateral sequence of operators in a Hilbert space* $\mathscr{H}$, *the following conditions are equivalent;*

(a) $\sum_{m,n} (T_{m-n} x_m, x_n) \geq 0$ *for every finitely nonzero bilateral sequence* (x_n) *of vectors in* $\mathscr{H}$.

(b) $\sum_{m,n} (T_{m-n} x, x) c_m \bar{c}_n \geq 0$ *for every* x *in* $\mathscr{H}$ *and every finitely nonzero bilateral sequence* (c_n) *of complex numbers.*

(c) *There exists an operator measure* F *(necessarily unique) on the Borel sets of the unit circle* $K = \{\lambda: |\lambda| = 1\}$ *such that*

$$T_n = \int \lambda^n dF$$

for all integers n.

A fundamental theorem of Nagy asserts that if T is any contraction operator (that is, $\|T\| \leq 1$), then the bilateral sequence $(T^{(n)})$ is of positive type [23, p. 106], where the operators $T^{(n)}$ are defined as follows:

$$T^{(n)} = \begin{cases} T^n & \text{for } n > 0, \\ I & \text{for } n = 0, \\ (T^*)^{|n|} & \text{for } n < 0. \end{cases}$$

It follows from Theorem 4 that if T is a contraction, there exists a unique operator measure F on the Borel sets of K such that

$$T^{(n)} = \int \lambda^n dF \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{14}$$

Such an operator measure is necessarily normalized (take $n = 0$ in (14)). Since $\lambda^{-1} = \bar{\lambda}$ for λ in K, the condition (14) is equivalent to

$$T^n = \int \lambda^n dF \quad (n = 1, 2, 3, \cdots) \tag{15}$$

(see [2, p. 27, Theorem 10]), and (15) implies

$$\int \lambda^n dF = \left(\int \lambda dF \right)^n \quad (n = 2, 3, 4, \cdots). \tag{16}$$

Definition 11. A *Nagy operator measure* is a normalized operator measure F on the Borel sets of the unit circle K that satisfies the condition (16).

The foregoing remarks constitute a proof of Nagy's fundamental representation theorem for contractions, which is a direct generalization of the spectral theorem for a unitary operator (see [22] and [20, p. 581]):

If T *is a contraction, there exists one and only one Nagy operator measure* F *such that* $T = \int \lambda dF$.

With notation as in the preceding statement, we call F the Nagy operator measure *associated* with the contraction T. In the reverse direction:

THEOREM 5. *If* F *is any operator measure on the Borel sets of the unit circle* K *such that* $F(K) \leq I$, *then the operator* $T = \int \lambda dF$ *is a contraction.*

This theorem follows immediately from the corollary of Theorem 1. For the case where F is normalized, it was proved by M. Schreiber [20, p. 581]. (Warning: even when $F(K) = I$, the Nagy operator measure associated with T does not coincide with F, unless F already satisfies the condition (16).) The proof of Theorem 5 is just as dependent on dilation theory as the corollary of Theorem 1; it would be nice to have a more direct proof. (Perhaps this is asking for too much; Naǐmark's dilation theorem is, after all, a piece of noncommutative integration theory.)

Definition 12. If T is a contraction in the Hilbert space $\mathscr{H}$, a *Nagy unitary dilation* of T is a pair $(\mathscr{D}, U)$, where $\mathscr{D}$ is a Hilbert space containing $\mathscr{H}$ as closed linear subspace, U is a unitary operator in $\mathscr{D}$, the vectors $U^n x$ ($x \in \mathscr{H}$, $n = 0, \pm 1, \pm 2, \cdots$) are total in $\mathscr{D}$, and

$$T^n x = PU^n x \quad (n = 1, 2, 3, \cdots) \tag{17}$$

for all x in $\mathscr{H}$ (P being the projection of $\mathscr{D}$ on $\mathscr{H}$).

Such a pair exists, and it is uniquely determined up to a unitary equivalence that leaves the vectors of $\mathscr{H}$ fixed (see [24, p. 15, Theorem III]); with this reservation, $(\mathscr{D}, U)$ is called *the* Nagy unitary dilation of T. Nagy's original construction of $(\mathscr{D}, U)$ was essentially as follows [22, p. 88, Theorem I]: let F be the Nagy operator measure associated with T, let $(\mathscr{D}, E)$ be the Naǐmark minimal spectral dilation of $(\mathscr{H}, F)$, and define $U = \int \lambda dE$. Later Nagy simplified the construction by eliminating operator measures from the argument [23], and I. Halperin reduced the proof to elementary Hilbert space geometry [11, p. 565].

We remark that (17) implies the more comprehensive condition that

$$T^{(n)}x = PU^n x \qquad (n = 0, \pm 1, \pm 2, \cdots) \tag{18}$$

for all x in $\mathscr{H}$; this follows easily from the definition of $T^{(n)}$ and the relation $U^{-1} = U^*$.

The following point is of didactic interest. Suppose T is a contraction. Pursuing an idea of P. R. Halmos [8, p. 126], J. J. Schäffer [19] gave a simple matricial construction of a unitary dilation ($\mathscr{D}$, U) satisfying the condition (18) (but not necessarily the minimality condition). From (18) it follows immediately that $(T^{(n)})$ is of positive type; this argument is simpler than Nagy's proof, which involves integration theory (see [23, p. 106] and [24, p. 30]). Schäffer's construction is also adequate for the proof of von Neumann's fundamental theorem on spectral sets (see [22] and [24, p. 17]).

5. CONCLUDING REMARKS

If the correspondence $s \to T_s$ in Naǐmark's moment theorem is a unitary representation of G, then the family (T_s) is obviously of positive type (see the computation in the proof of Theorem 2), and the operator measure F is necessarily a spectral measure [1, p. 592]; thus Naǐmark's result includes as a special case a theorem proved independently by W. Ambrose [1] and R. Godement [7]:

SPECTRAL THEOREM. *If* G *is a locally compact abelian group with character group* X, *and* $s \to U_s$ *is a weakly continuous unitary representation of* G, *there exists exactly one regular weakly Borel spectral measure* E *on* X *such that* $U_s = \int \hat{s}\, dE$ *for all* s *in* G.

The first theorem of this type is due to M. H. Stone [21], who proved it for the group of real numbers (see also [18, Section 137]). The key to the extension of Stone's theorem to arbitrary locally compact abelian groups is the Weil-Raǐkov generalization of the theorems of Herglotz and Bochner (see [18, Section 140] and [13, p. 147]).

Finally, we note an alternate proof of the part "(a) implies (c)" of Naǐmark's moment theorem: to produce the operator measure F, dilate the given family (T_s) to a unitary representation by Nagy's Principal Theorem, apply the Spectral Theorem to represent the unitary dilation by means of a spectral measure E, then compress E back to the original Hilbert space. In effect, the labor is divided into two parts: the algebra is performed by Nagy's theorem, the harmonic analysis by the generalized Stone theorem.

Added in proof. I am indebted to Professor Nagy for informing me that an elementary direct proof of the corollary of Theorem 1 appears in the work of C. Foiaş [*Décompositions intégrales des familles spectrales et semi-spectrales en opérateurs qui sortent de l'espace hilbertien,* Acta Sci. Math. Szeged 20 (1959), 117-155]; by uniform approximation it is sufficient to consider the case where f is simple, and this case is easily deduced from inequality (2.3) on page 122 of Foiaş's article. This also yields a dilation-free proof of Theorem 5.

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