A STABILITY CONDITION FOR $y'' + p(x)y = 0$

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In a recent paper, A. C. Lazer [1] showed that if $p(x) > 0$, $p(x) \in C^3(a, \infty)$, $p(x) \to +\infty$ as $x \to \infty$, and

$$\int_a^\infty |(p^{-1/2}(x))'''|\, dx < +\infty, \tag{1}$$

then all solutions of the equation

$$y'' + p(x)y = 0 \tag{2}$$

satisfy the condition

$$\lim_{x \to \infty} y(x) = 0. \tag{3}$$

In this note, we establish the same conclusion under weaker hypotheses for the case when p is monotonic.

THEOREM 1. *If* $p'(x) \geq 0$ *for* $a < x < \infty$, $p(x) \in C^3(a, \infty)$,

$$\lim_{x \to \infty} p(x) = +\infty, \tag{4}$$

and

$$\int_a^w |(p^{-\alpha}(x))'''|\, dx = o(p^{1-\alpha}(w)) \qquad (w \to \infty) \tag{5}$$

for some α $(0 < \alpha < 1)$, *then every solution of* (2) *satisfies* (3).

Proof. Writing $y^2 + p^{-1} y'^2 = v$ and $p^{-\alpha} = \phi$, we can easily verify the identity

$$\frac{d}{dx}\left\{p^{1-\alpha} v + \frac{1}{2}\phi'' y^2 - \phi' y' y\right\} = \frac{1}{2}\phi''' y^2 + (1 - 2\alpha) p' p^{-\alpha} y^2. \tag{6}$$

Now we observe that $v' = -p' p^{-2} y^2 \leq 0$; thus $v(x)$ is a positive, nonincreasing function. Therefore

$$\lim_{x \to \infty} v(x) = s \tag{7}$$

exists. If $s = 0$, then (3) clearly follows. We shall show that the assumption $s > 0$ yields a contradiction. It follows from (4) that all solutions of (2) are oscillatory; hence there exists a sequence $\{x_n\}$ such that $x_n \to \infty$ and

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(8) $$y'(x_n) = 0 \qquad (n = 1, 2, \cdots).$$

If $\varepsilon > 0$, then by (4) and (7) we may choose $b > a$ such that

(9) $$p(x) > 0 \quad \text{and} \quad s \leq v(x) < (1+\varepsilon)s$$

for $x \geq b$. Integrating (6) from b to x_n $(x_n > b)$, we obtain the relation

(10) $$p^{1-\alpha}(x_n)v(x_n) = C_0 - \frac{1}{2}\phi''(x_n)y^2(x_n) + \frac{1}{2}\int_b^{x_n} \phi''' y^2 dx + (1 - 2\alpha)\int_b^{x_n} p'p^{-\alpha} y^2 dx,$$

where C_0 is constant. Using the inequality

$$|\phi''(x_n)| \leq |C_1| + \int_b^{x_n} |\phi'''|\, dx,$$

we see from (9) and (10) that

$$sp^{1-\alpha}(x_n) \leq |C_0| + \left(|C_1| + \int_b^{x_n} |\phi'''|\, dx\right)(1+\varepsilon)s$$

$$+ (1 - \alpha)^{-1} p^{1-\alpha}(x_n)(1+\varepsilon)s \cdot \max(1 - 2\alpha, 0).$$

By (4) and (5), it now follows that

$$1 \leq (1 - \alpha)^{-1}(1+\varepsilon)\max(1 - 2\alpha, 0) + o(1) \qquad (n \to \infty).$$

This yields the desired contradiction with any $\varepsilon > 0$ if $\alpha \geq 1/2$, and with $\varepsilon < \alpha(1 - 2\alpha)^{-1}$ if $\alpha < 1/2$.

The following theorem can be proved similarly.

THEOREM 2. *If condition* (5) *of Theorem* 1 *is replaced by*

(5') $$\int_a^w |(p^{-1}(x))'''|\, dx = o(\log p(w)) \qquad (w \to \infty),$$

then the conclusion (3) *still holds.*

Remark. It is easy to verify that the function

$$p(x) = \int_0^x (2 + \cos t^{5/4})\, dt$$

satisfies condition (5') but not (1).

REFERENCE

1. A. C. Lazer, *A stability condition for the differential equation* $y'' + p(x)y = 0$, Michigan Math. J. 12 (1965), 193-196.

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