LARGE SUBGROUPS AND SMALL HOMOMORPHISMS

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1. INTRODUCTION

Unless otherwise stated, any group considered in this paper will be assumed to be an additively written p-primary abelian group for some prime p. For the most part, we follow the notation and terminology of [1]. All topological references will be to the p-adic topology. In the first section, we apply the concepts of large subgroups and small homomorphisms to generalize results in [7] on direct decompositions. The notions introduced then lead to the construction of p-groups that are neither transitive nor fully transitive. In the second section, the consideration of nonsmall homomorphisms of closed p-groups provides the existence theorems that make the results of the first section meaningful. Our methods also yield a simplified construction of members of a most remarkable class of p-groups discovered by Pierce [9]. Finally, in the last section, we show that a large subgroup is "large" in the sense that its structure essentially determines the structure of the containing group.

Let G be a p-group. We define $p^{\alpha}G$ for all ordinals α as follows:

(1) $pG = \{x \in G: x = pg, g \in G\}$,

(2) $p^{\alpha}G = p(p^{\alpha-1}G)$ if $\alpha - 1$ exists, and

(3) $p^{\alpha}G = \bigcap_{\beta<\alpha} p^{\beta}G$ if α is a limit ordinal.

We shall often write G^1 for $p^{\omega}G = \bigcap_{n<\omega} p^n G$. If $x \in G$, we define the *height* $h_G(x)$ of x in G by

$$h_G(x) = \begin{cases} n \text{ if } x \in p^n G \text{ and } x \notin p^{n+1}G \text{ for the integer } n, \\ \infty \text{ if } x \in G^1. \end{cases}$$

If G is reduced, we also define the *generalized height* $h_G^*(x)$ of x by

$$h_G^*(x) = \begin{cases} \alpha \text{ if } x \neq 0 \text{ and } \alpha+1 \text{ is the first ordinal such that } x \notin p^{\alpha+1}G, \\ \infty \text{ if } x = 0. \end{cases}$$

With each $x \in G$ we associate its *Ulm sequence* $U_G(x) = (\alpha_0, \alpha_1, \cdots)$, where $\alpha_i = h_G(p^i x)$ for each i. In the same manner, if G is reduced, we associate with x its *generalized Ulm sequence* $U_G^*(x)$. The ordinary and generalized Ulm sequences are partially ordered in the obvious term-by-term fashion, that is, $U_G(x) \geq U_G(y)$ if and only if $h_G(p^i x) \geq h_G(p^i y)$ for all i. We assume, of course, that $\infty > \alpha$ for all ordinals α. The n*th Ulm invariant* of G is denoted by $f_G(n)$.

Received April 5, 1965.

This paper was written while the author was an ONR Postdoctoral Research Associate supported by Office of Naval Research Contract Nonr-4683(00), NR 043-321.

A subgroup L of the p-group G is said to be *large* if (1) L is fully invariant and (2) $\{L, B\} = G$ for each basic subgroup B of G. Large subgroups were introduced in [9], where it was shown, in particular, that to each large subgroup L of G there corresponds a strictly increasing sequence $n(L) = (n_0, n_1, \cdots)$ of nonnegative integers such that $L = \{x \in G: U_G(x) \geq n(L)\}$. A homomorphism ϕ of a p-group G into a p-group K is said to be a *small homomorphism* if ker ϕ contains a large subgroup of G. The set $\mathrm{Hom}_s(G, K)$ of all small homomorphisms of G into K is a subgroup of Hom(G, K), and the set $E_s(G)$ of all small endomorphisms is an ideal in the endomorphism ring E(G) of G (see [9]). Recall that the ring R_p of p-adic integers, considered in the obvious manner as operators on the p-group G, is precisely the center of E(G) when G is unbounded.

2. SMALL HOMOMORPHISMS

If $\mathscr{A}$ is a class of p-groups, a p-group G will be said to be *$\mathscr{A}$-thick* if $\mathrm{Hom}(G, A) = \mathrm{Hom}_s(G, A)$ for all $A \in \mathscr{A}$. A p-group will be said to be *thick* if it is Σ-thick, where Σ denotes the class of all p-groups that are direct sums of cyclic groups. A bounded p-group is obviously $\mathscr{A}$-thick regardless of the class $\mathscr{A}$, and a divisible group is $\mathscr{A}$-thick provided all groups in $\mathscr{A}$ are reduced. In the next section, we shall establish the existence of less trivial examples; in particular, we shall find a large class (including all closed p-groups) of thick groups. It is evident that a direct summand of an $\mathscr{A}$-thick group is $\mathscr{A}$-thick, and Theorem 5.7 in [9] implies that the class of $\mathscr{A}$-thick p-groups is closed under pure extensions.

Lemma 2.1 and Theorem 2.3 below are generalizations of Theorems 1 and 2 in [7].

LEMMA 2.1. *Let* $G = A + K$, *where* $A \in \mathscr{A}$ *and* K *is* $\mathscr{A}$*-thick. If* H *is an* $\mathscr{A}$*-thick pure subgroup of* G, *then* $H \cap L \subseteq K$ *for some large subgroup* L *of* G.

Proof. Let π be the projection of G onto A associated with the direct decomposition $G = A + K$, and set $\phi = \pi \mid H$. Then $\phi \in \mathrm{Hom}(H, A)$, and consequently ker ϕ contains some large subgroup M of H. Since H is pure in G, G contains a large subgroup L such that $M = H \cap L$; indeed, L is determined in G by the same sequence that determines M in H. It remains only to observe that ker $\phi = K \cap H$.

Two abelian groups G and K will be said to be *essentially isomorphic* if there exist bounded pure subgroups A and B of G and K, respectively, such that $G/A \cong K/B$; that is, two groups are essentially isomorphic if and only if they have isomorphic direct summands with bounded complements. The proof of the following lemma can be accomplished by standard techniques, and it is indeed obvious for the case of p-groups.

LEMMA 2.2. *The abelian groups* G *and* K *are essentially isomorphic if and only if there exist bounded groups* A *and* B *such that* $G + A \cong K + B$.

THEOREM 2.3. *If* $G = A + H = B + K$, *where* A *and* B *are in* $\mathscr{A}$ *and* H *and* K *are* $\mathscr{A}$*-thick, then* A *and* H *are essentially isomorphic to* B *and* K, *respectively.*

Proof. By Lemma 2.1, there exist large subgroups L_1 and L_2 of G such that $H \cap L_1 \subseteq K \cap L_1$ and $K \cap L_2 \subseteq H \cap L_2$. The intersection $L = L_1 \cap L_2$ is a large subgroup of G such that $H \cap L = K \cap L$. Since $H \cap L = K \cap L$ is a large subgroup of both H and K, we can write $H = A' + H'$ and $K = B' + K'$, where A' and B' are bounded and $H'[p] = (H \cap L)[p] = K'[p]$. But H' and K' are direct summands of G with the same socle and are therefore isomorphic. Thus, H and K are essentially isomorphic. Since H' and K' have the same socle, the sets of complementary direct summands for H' and K' are the same, and therefore

$$A + A' \cong G/H' \cong G/K' \cong B + B'.$$

The conclusion that A and B are essentially isomorphic follows from the preceding lemma.

Following Kaplansky [6], we call a reduced p-group G *fully transitive* [*transitive*] if for each pair of elements g_1 and g_2 in G with

$$U_G^*(g_1) \leq U_G^*(g_2) \qquad [U_G^*(g_1) = U_G^*(g_2)]$$

there exists an endomorphism [automorphism] ϕ of G such that $\phi(g_1) = g_2$. Countable reduced p-groups and p-groups without elements of infinite height are known to be both transitive and fully transitive, and Kaplansky has suggested that "it seems plausible to conjecture" that all reduced p-groups are such. However, we now have the following result.

THEOREM 2.4. *If* $G = H + K$, *where* $H^1 \cong K^1 \cong C(p)$, H/H^1 *is in* $\mathcal{A}$, *and* K/K^1 *is* $\mathcal{A}$*-thick, then* G *is neither transitive nor fully transitive—in fact,* K^1 *is a fully invariant subgroup of* G.

Proof. Suppose $K^1 = \{a\}$ and $H^1 = \{b\}$. Then $U^*(a) = U^*(b) = (\omega, \infty, \infty, \cdots)$, and it suffices to show that there exists no endomorphism ϕ of G such that $\phi(a) = b$. Indeed, we show that $\phi(a) \in K$, and hence $\phi(K^1) \subseteq K \cap G^1 = K^1$, for every endomorphism ϕ of G.

Suppose ϕ is an endomorphism of G, and let $\bar{\phi}$ be the endomorphism of $\bar{G} = G/G^1$ induced by ϕ; that is, let $\bar{\phi}(x + G^1) = \phi(x) + G^1$. Note that $\bar{G} = \bar{H} + \bar{K}$, where

$$\bar{H} = \{H, G^1\}/G^1 \cong H/H^1 \qquad \text{and} \qquad \bar{K} = \{K, G^1\}/G^1 \cong K/K^1.$$

Let $\psi = \bar{\phi} \mid \bar{K}$. Then we can write $\psi = \zeta + \beta$, where ζ is a homomorphism of $\bar{K}$ into $\bar{H}$ and β is an endomorphism of $\bar{K}$. Since ζ is necessarily small, its kernel contains a large subgroup of $\bar{K}$ determined by some sequence $(n_0, n_1, \cdots)$. Choose $k \in K$ so that $p^{n_0+1}k = a$, and let $x = p^{n_0}k$. Then $px = a$ and $x + G^1 \in \ker \zeta$. Thus,

$$\phi(x) + G^1 = \bar{\phi}(x + G^1) = \beta(x + G^1) = y + G^1$$

for some $y \in K$. Therefore $\phi(x) - y \in G^1 \subseteq G[p]$ and $\phi(a) = \phi(px) = py \in K$.

The existence of a pair of groups H and K as in Theorem 2.4 is implied by Theorem 3.5 below.

From the foregoing theorem, it is evident that the well-known characterization given by Kaplansky [6] for the fully invariant subgroups of fully transitive p-groups cannot be extended to all p-groups. Also, contrary to the situation for countable p-groups (see [2]), there does indeed exist an uncountable p-group G and an ordinal α (namely, $\alpha = \omega$) such that $p^\alpha G$ has automorphisms not induced by automorphisms of G.

A perhaps even more striking example of a p-group that is neither transitive nor fully transitive is implicit in the following observation (the proof of which is quite similar to that of Theorem 2.4): If G is a p-group such that

$$E(G/G^1) = E_s(G/G^1) + R_p$$

(see [9] and Theorem 3.7 below) and G^1 is elementary, then every subgroup of G^1 is a fully invariant subgroup of G.

3. HOMOMORPHISMS OF CLOSED p-GROUPS

A p-group without elements of infinite height is said to be a *closed* p-*group* if each of its Cauchy sequences consisting of elements uniformly bounded in order has a limit in the p-adic topology; that is, a closed p-group is the torsion subgroup of its p-adic completion.

THEOREM 3.1. *Suppose* G *is an unbounded closed* p-*group, and let* K *be a* p-*group without elements of infinite height. Then* Hom(G, K) *properly contains* $\mathrm{Hom}_s(G, K)$ *if and only if* K *contains an unbounded closed* p-*group as a subgroup.*

Proof. If K does contain an unbounded closed p-group as a subgroup, then it is easy to exhibit a nonsmall homomorphism of G into K.

Suppose conversely that there exists a nonsmall homomorphism ϕ of G into K. Then ker ϕ contains no large subgroup of G, and by Corollary 2.10 in [9] we can find a positive integer k_0 and a sequence $x_1, x_2, \cdots$ of elements in G such that for each i,

$$0(x_i) \le p^{k_0}, \quad h_G(x_{i+1}) > \max[k_0 + h_G(x_i), h_K(\phi(x_i))], \qquad \phi(x_i) \ne 0, \quad \phi(x_i) \in K[p].$$

Moreover, it is not difficult to see that each x_i can be chosen to lie in a pure cyclic subgroup of G. Let $m_i = h_G(x_i)$, and choose c_i in G so that $p^{m_i} c_i = x_i$. It is then easy to see that

$$C = \{c_1, c_2, \cdots\} = \sum_{i=1}^{\infty} \{c_i\}$$

is a pure subgroup of G and that the $y_i = \phi(x_i)$ form a linearly independent subset of K[p]. Also, one can verify that $C \cap \ker \phi = \sum_{i=1}^{\infty} \{px_i\}$.

Let H be the closure of C in G, and set $C' = \phi(C)$ and $H' = \phi(H)$. Then H is an unbounded closed p-group, and we wish to show that the same is true of H'. Now

$$C' \cong C/C \cap \ker \phi \cong \sum_{i=1}^{\infty} C(p^{m_i+1}),$$

and H'/C' is divisible. In order to see that C' is a basic subgroup of H', it remains only to show that C' is pure in H'. However, this will follow readily from the observation that $H \cap \ker \phi$ is precisely the closure in G of $C \cap \ker \phi$. Then, to complete the proof that H' is a closed p-group, it suffices to show that every Cauchy sequence (relative to the p-adic topology on H') in $C'[p] = \sum_{i=1}^{\infty} \{y_i\}$ has a limit in H'. It is evident that every Cauchy sequence $z_1', z_2', \cdots$ in $C'[p]$ is the term-by-term image under ϕ of a Cauchy sequence $z_1, z_2, \cdots$ in H with each z_n contained in the bounded subgroup $\sum_{i=1}^{\infty} \{x_i\}$. Consequently the sequence $z_1, z_2, \cdots$ has a limit z in H, and $z' = \phi(z)$ is the limit in H' of the sequence $z_1', z_2', \cdots$. Thus, H' is a closed p-group, and it is necessarily unbounded since C' is unbounded.

COROLLARY 3.2. *If* G *is an unbounded closed* p-*group and* K *is a dense, pure subgroup of a closed* p-*group* $\overline{B}$ *such that* Hom(G, K) *properly contains* $\mathrm{Hom}_s(G, K)$, *then* $H[p] \subseteq K$ *for some unbounded direct summand* H *of* $\overline{B}$.

Proof. Let C' be as in the proof of Theorem 3.1, and choose a pure subgroup A of K such that $A[p] = C'[p]$. Then if M is the closure of A in $\overline{B}$, M is an unbounded direct summand of $\overline{B}$. If H' is also as in the proof of Theorem 3.1, we easily see that $M[p] \subseteq H'$.

A reduced p-group G will be said to be *quasi-closed* if the closure in G of every pure subgroup is again a pure subgroup (see [4]). By a *strictly quasi-closed* p-*group* we mean a quasi-closed p-group that is not a closed p-group. Closed p-groups are quasi-closed; whereas unbounded direct sums of cyclic groups are not. If a reduced primary group G is quasi-closed, then clearly $G^1 = 0$ and therefore G can be imbedded as a pure and dense subgroup of a closed p-group $\overline{B}$. If G is strictly quasi-closed, then $G \neq \overline{B}$, and by a remark in [4], $H[p] \not\subseteq G$ whenever H is an unbounded direct summand of $\overline{B}$. In fact, if $\overline{B}/G \cong C(p^\infty)$ (see Theorem 3.7 below) the condition that $H[p] \not\subseteq G$ whenever H is an unbounded direct summand of $\overline{B}$ is also sufficient for G to be quasi-closed. A primary group G satisfying this condition and such that $\overline{B}/G \cong C(p^\infty)$ was first constructed by Beaumont and Pierce in [10]. From Corollary 3.2 and these observations, we obtain the following.

COROLLARY 3.3. *If* G *is a closed* p-*group and* K *is a strictly quasi-closed* p-*group, then* $\mathrm{Hom}(G, K) = \mathrm{Hom}_s(G, K)$; *that is, if* $\mathcal{Q}$ *is the class of all strictly quasi-closed* p-*groups, then all closed* p-*groups are* $\mathcal{Q}$-*thick.*

In connection with Problem 13 in [1], the following may be of some interest.

THEOREM 3.4. *Let* B *be a basic subgroup of the closed* p-*group* $\overline{B}$. *If* $\overline{B} \sim G$, *then either* $B \sim G/G^1$, *or* G/G^1 *contains an unbounded closed* p-*group as a subgroup.*

Proof. If $\overline{B} \sim G$, then there exists a homomorphism ϕ of $\overline{B}$ onto G/G^1. If G/G^1 does not contain an unbounded closed p-group as a subgroup, then ϕ is necessarily small. In this case, $\overline{B}$ contains a large subgroup L such that G/G^1 is a homomorphic image of $\overline{B}/L = \{B, L\}/L \cong B/B \cap L$ and thus a homomorphic image of B itself.

THEOREM 3.5. *Let* G *be a dense pure subgroup of a closed* p-*group* $\overline{B}$. *If either* $G = \overline{B}$ *or* $|\overline{B}/G| = \aleph_0$, *then* G *is thick.*

Proof. If $G = \overline{B}$, then the conclusion is an immediate corollary of Theorem 3.1. Suppose $|\overline{B}/G| = \aleph_0$, and let ϕ be a homomorphism of G into a direct sum of cyclic groups. Then $A = \mathrm{im}\,\phi$ is a direct sum of cyclic groups, and A is the basic subgroup of a closed p-group $\overline{A}$. ϕ then extends uniquely to a homomorphism $\overline{\phi}$ of $\overline{B}$ into $\overline{A}$. If $H = \mathrm{im}\,\overline{\phi}$, then $|H/A| \leq \aleph_0$. Since A is a direct sum of cyclic groups and $H^1 = 0$, H is itself necessarily a direct sum of cyclic groups. Thus, $\overline{\phi}$ is small, and consequently $\phi = \overline{\phi} \mid G$ is also small.

COROLLARY 3.6. *There exists a* p-*group of length* $\omega + 1$ *that is neither transitive nor fully transitive.*

Proof. Choose p-groups H and K such that $H^1 \cong K^1 \cong C(p)$, H/H^1 is a direct sum of cyclic groups, and K/K^1 is a closed p-group. Then $H + K$ has length $\omega + 1$, and by Theorems 3.5 and 2.4, it is neither transitive nor fully transitive.

We close this section with two applications of Corollary 3.3. First, in answer to a question raised in [5], we show that there exists a p-group G such that no high subgroup of G is an endomorphic image of G. Let $\overline{B}$ be a closed p-group with a countable basic subgroup B, and let K be a quasi-closed p-group that is a proper pure, dense subgroup of $\overline{B}$. We can then construct (see [8]) a p-group G containing K as a high subgroup and such that $G/G^1 \cong \overline{B}$. Since all high subgroups of G can be identified with certain dense, pure subgroups of G/G^1 having the same socle, it follows from Corollary 1 in [4] that all high subgroups of G are strictly quasi-closed p-groups. Now each endomorphism of G into a high subgroup H induces a homomorphism ϕ of G/G^1 into H. By Corollary 2.3, ϕ is small. Thus, as in the proof of Theorem 3.4, $B \sim \operatorname{im} \phi$. But then $|\operatorname{im} \phi| \leq \aleph_0$ and $H \neq \operatorname{im} \phi$, since $|H| > \aleph_0$.

Our second application is a simple construction of an unbounded p-group such that $E(G) = E_s(G) + R_p$. The existence and remarkable properties of such groups were first established by Pierce in [9].

THEOREM 3.7. *Let* G *be a pure subgroup of the closed* p-*group* $\overline{B}$. *If* $\overline{B}/G \cong C(p^\infty)$ *and* G *is quasi-closed, then* $E(G) = E_s(G) + R_p$.

Proof. It follows from Theorem 7.5 in [9] that $\operatorname{Hom}_s(G, G) + R_p$ is a direct summand of $\operatorname{Hom}(G, G)$. Therefore it suffices to show that $\operatorname{Hom}(G, G)/\operatorname{Hom}_s(G, G)$ is a cyclic module over the ring of p-adic integers. From the pure exact sequence $0 \to G \to \overline{B} \to C(p^\infty) \to 0$ (see [3]), we obtain the exact sequences

$$0 \to \operatorname{Hom}(\overline{B}, G) \to \operatorname{Hom}(G, G) \to \operatorname{Pext}(C(p^\infty), G),$$

$$0 \to \operatorname{Hom}(C(p^\infty), C(p^\infty)) \to \operatorname{Pext}(C(p^\infty), G) \to 0.$$

From the latter sequence, it follows that $\operatorname{Pext}(C(p^\infty), G)$ is a cyclic module. By Corollary 3.3, $\operatorname{Hom}(\overline{B}, G) = \operatorname{Hom}_s(\overline{B}, G)$, and the proof is now completed by the observation that the image of $\operatorname{Hom}(\overline{B}, G)$ in $\operatorname{Hom}(G, G)$ is just $\operatorname{Hom}_s(G, G)$.

4. LARGE SUBGROUPS

We conclude this paper by showing that a large subgroup of a p-group is also large in the sense that, along with a certain finite number of the Ulm invariants, it determines the structure of the containing p-group. More precisely, we shall prove the following theorem, despite its lack of elegance.

THEOREM 4.1. *Let* L *be a large subgroup of a* p-*group* G *determined by the sequence* $(n_0, n_1, \cdots)$. *Suppose* ϕ *is a monomorphism of* L *into a* p-*group* K. *Then* ϕ *can be extended to an isomorphism of* G *onto* K *if and only if the following conditions are satisfied:*

(i) $f_G(n) = f_K(n)$ *for* $n \leq n_0$,

(ii) $h_G(x) = h_K(\phi(x))$ *for all* $x \in L$,

(iii) $\phi(L)$ *is a large subgroup of* K.

Proof. The three conditions are clearly necessary; let us therefore assume that they are satisfied. The group G contains a basic subgroup $B = \sum_{n=1}^{\infty} B_n$, where for each n either $B_n = 0$ or $B_n \cong \Sigma C(p^n)$. There exists a nondecreasing sequence $k_1, k_2, \cdots$ of nonnegative integers, with $k_n \leq n - 1$ for $n > n_0$ and such that

$$B \cap L = \sum_{n=n_0+1}^{\infty} p^{k_n} B_n.$$

Because of (i), we may assume that $B_n = 0$ for $n \leq n_0$. Now suppose $B_n = \Sigma\{b_\lambda^{(n)}\}$, and let $x_\lambda^{(n)} = p^{k_n} b_\lambda^{(n)}$. Then

$$p^{k_n} B_n = \Sigma\{x_\lambda^{(n)}\} \quad \text{and} \quad \phi(p^{k_n} B_n) = \Sigma\{y_\lambda^{(n)}\},$$

where $y_\lambda^{(n)} = \phi(x_\lambda^{(n)})$ and $h_K(y_\lambda^{(n)}) = k_n$. Choose $c_\lambda^{(n)} \in K$ such that $p^{k_n} c_\lambda^{(n)} = y_\lambda^{(n)}$, and set $C_n = \Sigma\{c_\lambda^{(n)}\}$. Setting $C = \sum_{n=n_0+1}^\infty C_n$, we see that there is an isomorphism ψ of B onto C such that

$$\psi(b_\lambda^{(n)}) = c_\lambda^{(n)} \quad \text{and} \quad \psi \mid B \cap L = \phi \mid B \cap L.$$

Clearly, C is pure in K. Since $\phi(L)$ is fully invariant in K, $\phi(L) \cap C_n = p^m C_n$ for some $m \leq k_n$. It can easily be shown that the assumption $m < k_n$ violates (ii). Thus,

$$\phi(L) \cap C = \sum_{n=n_0+1}^{\infty} \phi(L) \cap C_n = \sum_{n=n_0+1}^{\infty} p^{k_n} C_n = \phi(B \cap L).$$

It can also be shown that C is a basic subgroup of K. For if C is not a basic subgroup of K, then there exists an integer n and a $y \in K$ having order p^n such that $C_n + \{y\}$ is a direct summand of K. However, the existence of such a y can also be shown to contradict (ii).

We then define a mapping $\bar{\phi}$ of $G = \{L, B\}$ onto $K = \{\phi(L), C\}$ as follows:

$$\bar{\phi}(\ell + b) = \phi(\ell) + \psi(b) \quad (\ell \in L,\ b \in B).$$

It is then easy to verify that $\bar{\phi}$ is a well-defined homomorphism, and that it is one-to-one. $\bar{\phi}$ is then an isomorphism of G onto K such that $\bar{\phi} \mid L = \phi$.

COROLLARY 4.2. *Each automorphism of a large subgroup that preserves heights (as computed in the containing group) is induced by an automorphism of the containing* p-*group. In particular, every automorphism of* $p^n G$, *where* n *is an integer, is induced by an automorphism of* G.

COROLLARY 4.3. *If* H *and* K *are dense, pure subgroups of a* p-*group* G *such that* $H \cap L = K \cap L$ *for some large subgroup* L *of* G, *then* $H \cong K$.

Finally, we comment that Corollary 4.3, and hence Theorem 4.1 itself, is suggested by the proof of Theorem 2.3 above.

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