THE EMBEDDINGS $O(n) \subset U(n)$ AND $U(n) \subset Sp(n)$, AND A SAMELSON PRODUCT

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In this paper we discuss the embeddings of the orthogonal group $O(n)$ in the unitary group $U(n)$ and of $U(n)$ in the symplectic group $Sp(n)$ induced by the embeddings $R \subset C$ and $C \subset H$, where R, C, and H are the fields of real numbers, complex numbers, and quaternions, respectively. We find that these monomorphisms are homotopic to a composite mapping

$$O(n) \xrightarrow{\phi} U(n-1) \xrightarrow{j} U(n),$$

where ϕ is an analytic function (but not a homomorphism) and j is the usual inclusion. A similar result holds for the embeddings $U(n) \to Sp(n)$, and we prove it simultaneously.

A more detailed study of the map ϕ: $O(n) \to U(n-1)$ yields a further deformation $\tilde{\phi}$: $O(n) \to U(2[(n-1)/2])$, which is a "best possible" factorization (here [k] denotes the greatest integer in k). As we specialize further, the map ϕ: $O(2n+1) \to U(2n)$ induces a map ϕ': $V_{2n+1,2} \to S_{4n-1}$, which in turn induces the classical $\mathscr{C}_2$-isomorphism $\pi_k(V_{2n+1,2}) \approx \pi_k(S_{4n-1})$, where $\mathscr{C}_2$ is the class of 2-primary abelian groups. The map ϕ' can be made to yield some information on the 2-primary component of $\pi_k(V_{2n+1,2})$. We conjecture the existence of a similar map

$$O(2n+1) \to Sp(n)$$

that induces the $\mathscr{C}_2$-isomorphism $\pi_k(O(2n+1)) \approx \pi_k(Sp(n))$ of Harris [6], but work on this is incomplete. A more detailed study of the maps ϕ for the embeddings $U(n) \subset Sp(n)$ should yield results on this conjecture.

Finally, we use our results to calculate the order of the Samelson product $\langle \partial \iota_{2n}, \partial \iota_{2n} \rangle \in \pi_{4n-2}(O(2n))$, where ∂ is the transgression operator in the homotopy sequence of the fibration $O(2n+1) \to S_{2n}$. We are able to do this up to a factor of 2 for all n; and for $n \leq 4$, there is now sufficient knowledge of the homotopy groups of the appropriate Stiefel manifolds to calculate the exact order of this product. We are informed that by using entirely different methods, Mahowald [10] has calculated the order of $\langle \partial \iota_{2n}, \partial \iota_{2n} \rangle$ for $n \geq 4$.

1. NOTATION

Let F denote R, C, or H, where R is the field of real numbers, C is the field of complex numbers, and H is the field of quaternions. We use d for the dimension of F as an algebra over R, so that d = 1, 2, or 4. By F^n we denote an n-dimensional right vector space over F with a fixed basis and the usual inner product*

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$$\langle x, y\rangle = \langle (x_1, \cdots, x_n), (y_1, \cdots, y_n)\rangle = \sum_1^n \bar{x}_k y_k,$$

where $\bar{x}$ denotes the conjugate of x in F. We use O_n to denote the subgroup of GL(n, F) that preserves this inner product. Since we have fixed a basis for F^n, we are given a definite matrix representation of the group O_n, and it is easily seen that if $g \in O_n$ then g^{-1} is the conjugate transpose matrix, which we denote by ${}^t\bar{g}$. Since O_n is a group of linear transformations and F^n is a right vector space, O_n operates on the left, and we should regard $x = (x_1, \cdots, x_n) \in F^n$ as a column vector. In any case, O_n will be one of the groups O(n), U(n), or Sp(n), depending on the field F.

The inclusion mappings $i: F^{n-k} \to F^n$ defined by

$$i(x_1, \cdots, x_{n-k}) = (x_1, \cdots, x_{n-k}, 0, \cdots, 0)$$

induce monomorphisms $i: O_{n-k} \to O_n$, and we denote the right coset space $O_n/i(O_{n-k})$ by $O_{n,k}$. The spaces $O_{n,k}$ are homeomorphic to the Stiefel manifolds of k-frames in F^n, and we make this identification. The various mappings between these coset spaces yield the well-known fibre bundles $(O_{n,k}, p, O_{n,\ell}, O_{n-\ell,k-\ell})$, and we recall that $O_{n,1}$ is the (dn - 1)-sphere S_{dn-1}, while $O_{n,n} = O_n$, if we let O_0 denote the subgroup of O_1 consisting of the identity element.

Finally, the inclusions $R \subset C$ and $C \subset H$ induce monomorphisms

$$\alpha_C: O(n) \to U(n) \quad \text{and} \quad \alpha_H: U(n) \to Sp(n),$$

which we denote collectively by $\alpha_F: O_n \to O_n'$.

2. THE MAPS α AND THEIR DEFORMATIONS

In this section we define maps $\phi_C: O(n) \to U(n - 1)$ and $\phi_H: U(n) \to Sp(n - 1)$, and we discuss some of their properties.

Let $X_u(n) = \{ [x_{pq}] \in O_n |\ x_{nn} \neq u\}$, where $u \in F$ and $|u| = 1$.

PROPOSITION 2.1. *For each* $n \geq 2$, *there is a map* $X_u(n) \to O_{n-1}$.

Proof. According to our conventions, the projection $\pi: O_n \to S_{dn-1}$ picks out the last column of a matrix in O_n; that is, $\pi([x_{pq}]) = (x_{1n}, x_{2n}, \cdots, x_{nn})$. Thus $\pi: X_u(n) \to S_{dn-1} - \{u\}$, where we use π to denote the restriction of the projection map. Since $S_{dn-1} - \{u\}$ is contractible, there exists a cross section

$$\sigma_u: S_{dn-1} - \{u\} \to O_n.$$

For $g \in X_u(n)$, set

$$\phi_u(g) = [\sigma_u \pi(g)]^{-1} g.$$

Since $\pi(g) = \pi \sigma_u \pi(g)$ and π is the projection of a principle bundle, $\phi_u(g) \in O_{n-1}$.

COROLLARY 2.2. *For each* $n \geq 2$, *there exist maps*

$$\phi_C: O(n) \to U(n - 1) \quad \textit{and} \quad \phi_H: U(n) \to Sp(n - 1).$$

Proof. For $n \geq 2$, $\alpha_C: O(n) \to X_i(n) \subset U(n)$, where $i \in C$, and

$$\alpha_H: U(n) \to X_j(n) \subset Sp(n),$$

where $j \in H$. Thus we may define $\phi_C = \phi_i \alpha_C$ and $\phi_H = \phi_j \alpha_H$.

Let $j: X_u(n) \to O_n$ be the inclusion map.

PROPOSITION 2.3. *If* $i: O_{n-1} \to O_n$ *is the inclusion homomorphism, there exists a homotopy* $j \sim i\phi_u: X_u(n) \to O_n$ *for* $n \geq 2$.

Proof. Choose a deformation retraction r of $S_{dn-1} - \{u\}$ onto $\{-u\}$, say $r(x, 0) = x$, $r(x, 1) = -u$. Now define $\Phi_u: X_u(n) \times I \to O_n$ by

$$\Phi_u(g, t) = \theta(t)[\sigma_u r(\pi(g), t)]^{-1} g,$$

where θ is a path from the identity of O_n to $\sigma_u(-u)$. (We must choose $\sigma_u(-u)$ in the component of the identity of O_n in the real case.) Then $\Phi_u(g, 0) = \phi_u(g)$ and $\Phi_u(g, 1) = g$.

COROLLARY 2.4. *If* $i: O'_{n-1} \to O'_n$ *is the inclusion homomorphism, there exists a homotopy* $\alpha_F \sim \phi_F: O_n \to O'_n$ *for* $n \geq 2$.

Proof. Define $\Phi_C(g, t) = \Phi_i(\alpha_C(g), t)$ and $\Phi_H(g, t) = \Phi_j(\alpha_H(g), t)$.

For subsequent calculations, it will be useful to exhibit some actual formulae. As a cross-section $\sigma_u: S_{dn-1} - \{u\} \to O_n$ we may take

$$\sigma_u(x_1, x_2, \cdots, x_n) = \left[\begin{array}{ccc|c} & & & x_1 \\ & & & x_2 \\ & [\delta_{pq} - x_p Q^{-1}\bar{x}_q] & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & x_{n-1} \\ \hline P\bar{x}_1 & P\bar{x}_2 & \cdots\ P\bar{x}_{n-1} & x_n \end{array}\right],$$

where $Q = 1 - \bar{x}_n u$ and $P = u\bar{Q}Q^{-1}$. Since our projection $\pi: O_n \to S_{dn-1}$ is the map $\pi([x_{pq}]) = (x_{1n}, x_{2n}, \cdots, x_{nn})$, we easily calculate that for $1 \leq p, q \leq n$, $\phi_u([x_{pq}])$ is the $(n-1)$-by-$(n-1)$ matrix $[y_{rs}]$, where

$$y_{rs} = x_{rs} + x_{rn}(u - x_{nn})^{-1} x_{ns}$$

for $1 \leq r, s \leq n-1$. Of course, in the case of the maps ϕ_C and ϕ_H, u will be replaced by i and j, respectively.

PROPOSITION 2.5. (i) *If* $G \subset O_{n-1}$, *then* $\phi_F | G = \alpha_F | G$.

(ii) *If* $g \in O_{n-1}$, *then* $\phi_F(xg) = \phi_F(x)\alpha_F(g)$.

Proof. Part (i) follows easily from the formulae, because $x_{pn} = x_{nq} = 0$ for $1 \leq p, q \leq n-1$. To see part (ii), observe that if $g \in O_{n-1}$, then

$$\phi_u(xg) = [\sigma_u \pi(xg)]^{-1} xg = [\sigma_u \pi(x)]^{-1} xg = \phi_u(x) g.$$

Applying this to ϕ_F, we obtain the formula of (ii).

COROLLARY 2.6. *The maps* ϕ_F *induce bundle maps*

$$\phi'_F\colon (O_{n,k}, \pi, O_{n,\ell}, O_{n-\ell,k-\ell}) \to (O'_{n-1,k-1}, \pi', O'_{n-1,\ell-1}, O'_{n-\ell,k-\ell}) \quad \textit{for } 2 \leq \ell \leq k \leq n-1,$$

$$\phi'\colon (O_{n,k}, \pi, O_{n,\ell}, O_{n-\ell,k-\ell}) \to (O'_{n-1}/\alpha_F(O_{n-k}), \pi, O'_{n-1}/\alpha_F(O_{n-\ell}), O_{n-\ell,k-\ell}) \quad \textit{for } 1 \leq \ell \leq k \leq n.$$

Proof. By property (ii) of Proposition 2.5, ϕ_F induces maps on the appropriate coset spaces.

3. THE MAP O(n) → U(n)

From now on, we are concerned only with the maps derived from the inclusion $\alpha_C\colon O(n) \to U(n)$. We therefore drop the subscripts, writing α, ϕ, ϕ', $\cdots$ for the maps α_C, ϕ_C, ϕ'_C, $\cdots$, respectively. Also, since we are no longer concerned with the quaternions, multiplication of field elements is commutative.

LEMMA 3.1. *Let* $V_{2n+2,2}$ *and* $W_{2n+2,2}$ *be the Stiefel manifolds of orthonormal 2-frames in real and complex* $(2n+2)$*-space, respectively. If*

$$\alpha'\colon V_{2n+2,2} \to W_{2n+2,2}$$

is induced by the map α, *then* α' *is null-homotopic.*

Proof. Represent the points of $O_{2n+2,2}$ by the symbols $[y, x]$, where y and x are orthonormal vectors. Let

$$e = (0, 0, \cdots, 0, 1) \quad \text{and} \quad e' = (0, 0, \cdots, 1, 0),$$

and choose $[e', e]$ as the base point of $O_{2n+2,2}$. Since S_{2n+1} is an odd-dimensional sphere, we may define a function $\omega\colon S_{2n+1} \to S_{2n+1}$ by $\omega(x) = z$, where $z_{2k-1} = x_{2k}$ and $z_{2k} = -x_{2k-1}$ for $1 \leq k \leq n+1$. Note that if we define $\overline{\omega}(x) = [\omega(x), x]$, then $\overline{\omega}\colon (S_{2n+1}, e) \to (V_{2n+2,2}, [e', e])$ is a base-point-preserving cross-section of the tangent sphere bundle of S_{2n+1}. Define $F\colon V_{2n+2,2} \times I \to W_{2n+2,2}$ by

$$F([y, x], t) = \begin{cases} [\exp(\pi i t)(cy - is\omega(x)), x] & (0 \leq t \leq 1/2), \\ [\exp(\pi i t)(ce' - is\omega(x)), \exp(-\pi i t)(ce + isx)] & (1/2 \leq t \leq 1), \end{cases}$$

where $c = \cos(\pi t)$ and $s = \sin(\pi t)$. Note that

$$F([y, x], 0) = \alpha'([y, x]), \qquad F([y, x], 1) = [e', e],$$

$$F([e', e], t) = [e', e] \quad \text{for } 0 \leq t \leq 1.$$

Thus F is the desired homotopy.

For application in the proof of the following theorem it is useful to observe that

$$F([y, e], t) = [f(y, t), e] \in W_{2n+2,2}, \quad \text{where } f\colon S_{2n} \times I \to S_{4n+3} - \{i\}.$$

We use Lemma 3.1 to prove the following theorem, which sharpens Corollaries 2.2 and 2.4.

PROPOSITION 3.2. *For* $n \geq 0$, *there exists a map*

$$\tilde{\phi}: O(2n+2) \to U(2n)$$

such that if $j: U(2n) \to U(2n+2)$ *is the inclusion,* $j\tilde{\phi}$ *is homotopic to* α *relative to* $O(2n)$, *and* $\tilde{\phi} \mid O(2n+1)$ *is the map* ϕ, *described in Corollary* 2.2.

Proof. The method of proof is to extend the map in Corollary 2.2. Let

$$\pi: O(2n+2) \to V_{2n+2,2} \quad \text{and} \quad \pi': U(2n+2) \to W_{2n+2,2}$$

be the bundle projections. Define $F': O(2n+2) \times I \to W_{2n+2,2}$ by $F' = F(\pi \times 1)$. Then F' is a null-homotopy, by Lemma 3.1. Now define $G: O(2n+1) \times I \to U(2n+1)$ by

$$G(x, t) = A(x, t)[\sigma_i f(\pi''(x), 1 - t)]^{-1} \alpha(x),$$

where $\pi'': O_{2n+1} \to S_{2n}$ is the usual projection, $\sigma_i: S_{4n+1} - \{i\} \to U(2n+1)$ is the cross-section of Section 2, and $A(x, t)$ is the unitary transformation with matrix

$$\begin{bmatrix} -\exp(-2\pi it)\overline{P} & & & & 0 \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & -\exp(-2\pi it)\overline{P}' & 0 \\ 0 & & \cdots & 0 & 1 \end{bmatrix}$$

(here P comes from the formula for $\sigma_i f(\pi''(x), 1 - t)$ as in Section 2). We verify by direct calculation that

$$\pi' G = F' \mid O(2n+1),$$

$$G(x, 0) = \alpha(x), \quad G(x, 1) = \phi(x), \quad G(x, t) = \alpha(x) \quad \text{for } x \in O(2n).$$

Now extend G to

$$G': O(2n+2) \times \{0\} \cup O(2n+1) \times I \to U(2n+2)$$

by setting $G'(x, 0) = \alpha(x)$. By a covering homotopy extension theorem [1, Theorem 2.4], there is a map $\Phi: O(2n+2) \times I \to U(2n+2)$ such that Φ extends G' and $\pi'\Phi = F'$. Set $\tilde{\phi}(x) = \Phi(x, 1)$. Since $\pi'\tilde{\phi}(x) = F'(x, 1) = [e', e]$, $\tilde{\phi}(x) \in U(2n)$. The homotopy of $j\tilde{\phi}$ with α is given by Φ, and since Φ extends G', $\tilde{\phi} \mid O(2n+1) = \phi$.

Thus Propositions 2.2 and 2.4 are conveniently combined in the following theorem.

THEOREM 3.3. *For* $n \geq 2$, *there exists a map*

$$\tilde{\phi}: O(n) \to U(2[(n-1)/2])$$

such that if $j: U(2[(n-1)/2]) \to U(n)$ *is the inclusion map, then*

$$j\tilde{\phi} \sim \alpha \ (\text{rel } O(2[(n-1)/2])).$$

Proof. If n is odd, we take $\tilde{\phi} = \phi$. Otherwise, take $\tilde{\phi}$ to be the map in Proposition 3.2.

Consider the following sequences of bundles:

$$\xi^n_{k,\ell}\colon V_{n,k} \xrightarrow{\pi} V_{n,\ell} \text{ with fibre } V_{n-\ell,k-\ell} \qquad (1 \leq \ell \leq k \leq n),$$

$$\eta^n_{k,\ell}\colon W_{n,k} \xrightarrow{\pi'} W_{n,\ell} \text{ with fibre } W_{n-\ell,k-\ell} \qquad (1 \leq \ell \leq k \leq n),$$

$$\theta^n_{k,\ell}\colon U(n)/O(n-k) \xrightarrow{\pi} U(n)/O(n-\ell) \text{ with fibre } V_{n-k,k-\ell} \qquad (0 \leq \ell \leq k \leq n).$$

THEOREM 3.4. *The maps* $\tilde{\phi}$ *induce the following commutative diagrams of bundle maps.*

(i) *For* $2 \leq \ell \leq k \leq 2n+1$,

$$\begin{array}{ccc} \xi^{2n+1}_{k,\ell} & \xrightarrow{\ \phi'\ } & \eta^{2n}_{k-1,\ell-1} \\ & {\scriptstyle i'}\searrow \quad \nearrow{\scriptstyle \tilde{\phi}'} & \\ & \xi^{2n+2}_{k+1,\ell+1} & \end{array}$$

(ii) *For* $1 \leq \ell \leq k \leq 2n+1$,

$$\begin{array}{ccc} \xi^{2n+1}_{k,\ell} & \xrightarrow{\ \phi''\ } & \theta^{2n}_{k-1,\ell-1} \\ & {\scriptstyle i'}\searrow \quad \nearrow{\scriptstyle \tilde{\phi}''} & \\ & \xi^{2n+2}_{k+1,\ell+1} & \end{array}$$

Proof. As in Proposition 2.5, if $g \in O(2n)$, then $\tilde{\phi}(xg) = \tilde{\phi}(x)\alpha(g)$; thus $\tilde{\phi}$ induces maps on the appropriate coset spaces.

We want to investigate the effect of the map $\tilde{\phi}$ on homotopy. We recall some preliminary notions and introduce some notation. Let U and O denote the increasing union of the groups U(n) and O(n), respectively. According to Bott [4], $\pi_{2n-1}(U)$ is an infinite cyclic group generated by u^∞_{2n-1}; $\pi_{4n-1}(O)$ is an infinite cyclic group generated by o^∞_{4n-1}; and the maps $\alpha\colon O(n) \to U(n)$ induce a map $\alpha\colon O \to U$ such that

$$\alpha_*(o^\infty_{4n-1}) = a_n u^\infty_{4n-1},$$

where $a_n = 2$ if $n \equiv 1 \pmod 2$ and $a_n = 1$ if $n \equiv 0 \pmod 2$. The homotopy sequence of a fibration implies that for $k \geq n$, $\pi_{2n-1}(U(k))$ is infinite cyclic and is generated by an element u^k_{2n-1} such that under the inclusion $U(k) \to U$, $u^k_{2n-1} \to u^\infty_{2n-1}$; together with $\mathscr{C}$-theory, it shows that for $k \geq 2n+1$, $\pi_{4n-1}(O(k))$ has an infinite summand generated by o^k_{4n-1} such that under the inclusion $O(k) \to O$,

$$o^k_{4n-1} \to 2^{\beta(n,k)} o^\infty_{4n-1}.$$

Finally, $\beta(n, k)$ is completely known; it is given in the following table:

n \ k	$\geq 4n$	$4n - 1$	$4n - 2$	$4n - 3$	$4n - 4 \leq k \leq 2n + 1$
$n \neq 1,2,4$	0	0	0	0	0
1	0	1	--	--	--
2	0	1	2	3	--
4	0	0	0	0	1

The first column follows from the homotopy sequence of a fibration, the first row is due to Barratt and Mahowald [3], and the rest of the table follows from work of Kervaire [9].

THEOREM 3.5. *Let* $k \geq 2n + 1 \geq 3$, *and let* $m = 2[(k - 1)/2]$. *Then the homomorphism*

$$\tilde{\phi}_*: \pi_{4n-1}(O(k)) \to \pi_{4n-1}(U(m))$$

has the property $\tilde{\phi}_*(o^k_{4n-1}) = 2^{\beta(n,k)} a_n u^m_{4n-1}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} & & \pi_{4n-1}(O(k)) & \xrightarrow{i'_*} & \pi_{4n-1}(O) \\ & \swarrow^{\tilde{\phi}_*} & \downarrow \alpha_* & & \downarrow \alpha_* \\ \pi_{4n-1}(U(m)) & \xrightarrow{j_*} & \pi_{4n-1}(U(k)) & \xrightarrow{j'_*} & \pi_{4n-1}(U), \end{array}$$

where the horizontal maps are induced by inclusions. The triangle is commutative, because $j\tilde{\phi} \sim \alpha$. The homomorphisms on the bottom line are isomorphisms, because the groups are stable. Thus

$$2^{\beta(n,k)} a_n u^\infty_{4n-1} = \alpha_* i'_*(o^k_{4n-1}) = j'_* j_* \tilde{\phi}_*(o^k_{4n-1}) = j'_* j_*(2^{\beta(n,k)} a_n u^m_{4n-1}),$$

and since $j'_* j_*$ is an isomorphism, $\tilde{\phi}_*(o^k_{4n-1}) = 2^{\beta(n,k)} a_n u^m_{4n-1}$.

We may use this theorem, at least the nontriviality of $\tilde{\phi}_*$ or α_*, to prove that Theorem 3.3 is best possible in the following sense.

COROLLARY 3.6. *There exists no map* ψ: $O(n) \to U(r)$ *for* $r < 2[(n - 1)/2]$ *such that* $j\psi \sim \alpha$, *where* j: $U(r) \to U(n)$ *is the inclusion.*

Proof. Let $m = [(n - 1)/2]$. Then $\pi_{4m-1}(O(n)) \xrightarrow{\alpha_*} \pi_{4m-1}(U(2m))$ is nontrivial, but since $r < 2m$, $\pi_{4m-1}(U(r)) \xrightarrow{j_*} \pi_{4m-1}(U(2m))$ is the trivial map.

4. THE MAP $V_{n,2} \to S_{2n-3}$

We want to study the map ϕ': $V_{n,2} \to S_{2n-3}$ induced from the bundle map $\xi^n_{n,2} \to \eta^{n-1}_{n-1,1}$. We shall need some facts about $V_{n,2}$.

Recall that $V_{n,2}$ is the set of pairs $[y, x]$ of orthonormal vectors y and x in real n-space. The topology of $V_{n,2}$ is induced from the embedding of $V_{n,2}$ in (2n)-space: $[y, x] \to (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$. We define two "hemispheres" and an "equator" on $V_{n,2}$ as follows:

$$U^+_{n,2} = \{[y, x] \in V_{n,2} \mid y_n > 0\},$$

$$U^-_{n,2} = \{[y, x] \in V_{n,2} \mid y_n < 0\},$$

$$E_{n,2} = \{[y, x] \in V_{n,2} \mid y_n = 0\}.$$

Since the topoloty is induced from (2n)-space, it is clear that $U^+_{n,2}$ and $U^-_{n,2}$ are open and separated in $V_{n,2}$, and the intersection of their closures is $E_{n,2}$. Note also that $V_{n,2}$ and S_{2n-3} are both $(2n - 3)$-manifolds.

PROPOSITION 4.1. *The map* ϕ': $V_{n,2} \to S_{2n-3}$ *induced by* ϕ *has the property that* $\phi' \mid U^{\pm}_{n,2}$ *is a homeomorphism onto an open subset of* S_{2n-3}. *Moreover, for each* $[y, x] \in U^+_{n,2} \cup U^-_{n,2}$, *the set* $\phi'^{-1}\phi'([y, x])$ *consists of precisely two elements.*

Proof. From the formula following Corollary 2.4, it is easily verified that $\phi'([y, x]) = (u_1 + iv_1, u_2 + iv_2, \cdots, u_{n-1} + iv_{n-1}) \in S_{2n-3}$, where

$$u_p = y_p - (y_n x_n x_p)/(1 + x_n^2), \quad v_p = -(y_n x_p)/(1 + x_n^2), \qquad \text{for } 1 \leq p \leq n - 1.$$

Set $\triangle_x = 1 + x_n^2$, and observe that

$$\sum v_p^2 = y_n^2(1 - x_n^2)/\triangle_x^2 \quad \text{and} \quad \sum u_p v_p = 2x_n y_n^2/\triangle_x^2,$$

because $\sum x_k^2 = \sum y_k^2 = 1$ and $\sum x_k y_k = 0$. Now define

$$A^2 = \left(\sum v_k^2\right)^2 + \left(\sum u_k v_k\right)^2 = y_n^4/\triangle_x^2,$$

and set

$$B = \sum v_p^2 + A = 2y_n^2/\triangle_x^2.$$

Then

$$Bx_n = \sum u_p v_p, \quad By_n^2 = 2A^2, \quad Ax_p = -v_p y_n, \quad By_p = Bu_p - v_p \sum u_k v_k.$$

Observe that $B = 0$ if and only if $A = 0$, and that this is equivalent to the condition $y_n = 0$. Thus we can solve these equations for $[x, y]$ provided the vector $\phi'([x, y])$ is not a real vector. In the case $A \neq 0$, we easily find that

$$y_n = \lambda A\sqrt{2/B}, \quad x_n = \sum u_k v_k/B, \quad x_p = -\lambda v_p \sqrt{2/B}, \quad By_p = Bu_p - v_p\left(\sum u_k v_k\right),$$

where $\lambda = \pm 1$ and $1 \leq p \leq n - 1$. From this calculation it is clear that for $[y, x] \in U^+_{n,2} \cup U^-_{n,2}$, $\phi'^{-1}\phi'([y, x])$ consists of precisely two elements, corresponding to the choice of λ. If $W \subset S_{2n-3}$ is the set of complex vectors none of which has only real coordinates (this set is homeomorphic to $S_{2n-3} - S_{n-2}$), then the map ψ^+: $W \to U^+_{n,2}$ given by

$$\psi^+(u+iv) = [\psi_1^+(u+iv),\ \psi_2^+(u+iv)],$$

where

$$\psi_1^+(u+iv) = (u_1 - (v_1/B)\textstyle\sum u_k v_k, \cdots, u_{n-1} - (v_{n-1}/B)\sum u_k v_k,\ A\sqrt{2/B})$$

$$\psi_2^+(u+iv) = (-v_1\sqrt{2/B}, \cdots, -v_{n-1}\sqrt{2/B},\ \left(\textstyle\sum u_k v_k\right)/B),$$

and where A and B are defined as above, is an inverse for $\phi' \mid U^+_{n,2}$. From the formulae, it is clear that both ϕ' and ψ^+ and a similarly defined map ψ^- are (real) analytic maps.

THEOREM 4.2. *For* $n \geq 2$, *the map* $\phi'\colon V_{n,2} \to S_{2n-3}$ *has the following algebraic properties:*

(i) *degree* $\phi' = 0$ *if* n *is even,*

(ii) $|\text{degree } \phi'| = 2$ *if* n *is odd.*

Proof. Since ϕ' is a local diffeomorphism on $U^+_{n,2} \cup U^-_{n,2}$, the local degree of ϕ' at $[y, x] \in U^+_{n,2} \cup U^-_{n,2}$ is ± 1. By Proposition 4.1, the relation

$$\phi'^{-1}\phi'([y, x]) = \{[y, x], [y', x']\}$$

holds, on this set; therefore according to [11],

$$\text{degree } \phi' = \deg_{[y,x]} \phi' + \deg_{[y',x']} \phi'.$$

Thus degree $\phi' = 0$ or 2.

In the case where n is even, we know by Lemma 3.1 that α' is null-homotopic, where $\alpha'\colon V_{n,2} \to W_{n,2}$. Also, if $i'\colon S_{2n-3} \to W_{n,2}$ is the inclusion, then $i'\phi' \sim \alpha'$. Therefore, in homology, $i'_* \phi'_* = \alpha_* = 0$. Since $i'_*\colon H_{2n-3}(S_{2n-3}) \to H_{2n-3}(W_{n,2})$ is an isomorphism, $\phi'_* = 0$, and degree $\phi' = 0$.

In the case where n is odd, say $n = 2k+1$, consider the diagram

$$\begin{array}{ccc}
\pi_{4k-1}(O(2k+1)) & \xrightarrow{\phi_*} & \pi_{4k-1}(U(2k)) \\
\downarrow \pi_* & & \downarrow \pi'_* \\
\pi_{4k-1}(V_{2k+1,2}) & \xrightarrow{\phi'_*} & \pi_{4k-1}(S_{4k-1}) \\
\downarrow h & & \downarrow h' \\
H_{4k-1}(V_{2k+1,2}) & \xrightarrow{\phi'_*} & H_{4k-1}(S_{4k-1}),
\end{array}$$

where h is the Hurewicz homomorphism. By Theorem 3.5,

$$\phi_*(o^{2k+1}_{4k-1}) = 2^{\beta(k,2k+1)} a_k u^{2k}_{4k-1},$$

and by [5], $\pi'_*(u^{2k}_{4k-1}) = \pm(2k-1)!\,\iota_{4k-1}$, hence

$$\pi'_* \phi_*(o^{2k+1}_{4k-1}) = \pm 2^{\beta(k,2k+1)} a_k (2k-1)!\,\iota_{4k-1}$$

Now h' is an isomorphism, so that $h'\pi'_* \phi_* \neq 0$. By commutativity of the diagram, $\phi'_* h \pi_* \neq 0$, so that $\phi'_* \neq 0$. We conclude that $|\text{degree } \phi'| = 2$.

COROLLARY 4.3. *For* $n \geq 1$, *the map* $\phi'\colon V_{2n+1,2} \to S_{4n-1}$ *induces a* $\mathscr{C}_2$-*isomorphism* $\phi'_*\colon \pi_j(V_{2n+1,2}) \to \pi_j(S_{4n-1})$ *for all* j.

Proof. Since degree $\phi' = \pm 2$, $\phi'_*\colon H_j(V_{2n+1,2}) \to H_j(S_{4n-1})$ is a $\mathscr{C}_2$-isomorphism for all j. An application of Whitehead's theorem for $\mathscr{C}_2$ yields the result.

Since $\phi'_*\colon \pi_{4n-1}(V_{2n+1,2}) \to \pi_{4n-1}(S_{4n-1})$ is nontrivial, there exists a generator $\theta_{4n-1} \in \pi_{4n-1}(V_{2n+2,2})$ of infinite order.

THEOREM 4.4. *For* $n \geq 2$, $\phi'_*(\theta_{4n-1}) = \pm 8a_n \iota_{4n-1}$, *and* $\phi'_*(\theta_3) = \pm 4\iota_3$.

Proof. Let $v_{4n-1} \in H_{4n-1}(V_{2n+1,2})$ and $s_{4n-1} \in H_{4n-1}(S_{4n-1})$ be generators. For notation, we refer to the lower half of the diagram in the proof of Theorem 4.2. According to James [7, Theorem 7.3], for $n \geq 2$, $h(\theta_{4n-1}) = \pm 4a_n v_{4n-1}$, and $h(\theta_3) = \pm 2v_3$. Since ϕ' is of degree ± 2, we see that $\phi'_* h(\theta_{4n-1}) = \pm 8a_n s_{4n-1}$ and $\phi'_* h(\theta_3) = \pm 4s_3$. But h' is an isomorphism; hence $\phi'_*(\theta_{4n-1}) = \pm 8a_n \iota_{4n-1}$ for $n \geq 2$, and $\phi'_*(\theta_3) = \pm 4\iota_3$.

Let $E\colon \pi_k(S_n) \to \pi_{k+1}(S_{n+1})$ be the suspension homomorphism. We can sharpen Corollary 4.3 as follows.

COROLLARY 4.5. *For* $n \geq 2$, $8a_n E\pi_k(S_{4n-2}) \subset \phi'_* \pi_{k+1}(V_{2n+1,2})$, *and* $4E\pi_k(S_2) \subset \phi'_* \pi_{k+1}(V_{3,2})$.

Proof. If $8a_n E\lambda \in 8a_n E\pi_k(S_{4n-2})$, then

$$\phi'_*(\theta_{4n-1} \circ E\lambda) = (8a_n \iota_{4n-1}) \circ E\lambda = 8a_n(\iota_{4n-1} \circ E\lambda) = 8a_n E\lambda .$$

The proof for $n = 1$ is similar.

5. A SAMELSON PRODUCT

The first portion of this section will be concerned with the homotopy sequence of the bundle $(V_{2n+1,2}, p, S_{2n}, S_{2n-1})$. We use Δ to denote the homotopy transgression in this sequence.

LEMMA 5.1. *If* $\lambda \in \pi_{4n-1}(V_{2n+1,2})$ *is an element of finite order, then* $p_*(\lambda) \in \pi_{4n-1}(S_{2n})$ *is of order at most* 2.

Proof. Recall that $\pi_{4n-1}(S_{2n}) = Z + E\pi_{4n-2}(S_{2n-1})$, where E is the suspension. Thus $p_*(\lambda) = E\beta = \iota_{2n} \circ E\beta$, and

$$0 = \Delta p_*(\lambda) = \Delta(\iota_{2n} \circ E\beta) = \Delta\iota_{2n} \circ \beta = 2\iota_{2n-1} \circ \beta = 2(\iota_{2n} \circ \beta) = 2\beta,$$

by James [7, Lemma 3.6]. Since $p_*(\lambda)$ is the suspension of an element of order at most 2, $p_*(\lambda)$ is of order at most 2.

Let $\omega_{2n} = [\iota_{2n}, \iota_{2n}] \in \pi_{4n-1}(S_{2n})$ be the Whitehead square of $\iota_{2n} \in \pi_{2n}(S_{2n})$.

PROPOSITION 5.2. *For* $n \geq 2$, *there is a generator* $\theta_{4n-1} \in \pi_{4n-1}(V_{2n+1,2})$ *such that* $p_*(\theta_{4n-1}) = a_n \omega_{2n}$.

Proof. According to James [7, relation (7.4)], there exists an element θ_{4n-1} such that $p_*(\theta_{4n-1}) = a_n\omega_{2n}$. We want to show that θ_{4n-1} is a generator. Suppose that $n \neq 2, 4$, so that $\pi_{4n-1}(S_{2n})$ is generated by ω_{2n} and elements of finite order. Let $\theta' \in \pi_{4n-1}(V_{2n+1,2})$ be a generator of infinite order, let $p_*(\theta') = q\omega_{2n} + \beta$, where β is of finite order, and suppose that $\theta_{4n-1} = r\theta' + \alpha$, where α is of finite order. Then

$$a_n\omega_{2n} = p_*(\theta_{4n-1}) = rq\omega_{2n} + r\beta + p_*(\alpha),$$

and $rq = a_n$ and $r\beta = -p_*(\alpha)$. If $n \equiv 0 \pmod 2$, then $a_n = 1 = rq$, so that $|r| = |q| = 1$, and θ_{4n-1} is a generator. If $n \equiv 1 \pmod 2$, then $a_n = 2 = rq$. If $|r| = 2$, then $-2\beta = p_*(\alpha)$, and

$$0 = \Delta p_*(\theta') = \Delta(q\omega_{2n} + \beta) = [\iota_{2n-1}, \eta_{2n-1}] + \Delta\beta$$

$$= [\iota_{2n-1}, \eta_{2n-1}] + \Delta(\iota_{2n} \circ E\beta') = [\iota_{2n-1}, \eta_{2n-1}] + 2\beta',$$

where η_{2n-1} is the generator of $\pi_{2n}(S_{2n-1})$. By [7, Lemma 3.5], $[\iota_{2n-1}, \eta_{2n-1}] \neq 0$, so that $2\beta' \neq 0$. Thus $[\iota_{2n-1}, \eta_{2n-1}] = 2\beta' \in 2\pi_{4n-2}(S_{2n-1})$, which is impossible, by [7, Lemma 5.2]. We conclude that $|r| = 1$ and θ_{4n-1} is a generator.

In the cases $n = 2, 4$, $\pi_{4n-1}(S_{2n})$ is generated by an element γ_{2n} of infinite order and an element λ_{2n} of order

$$k = 12 \quad (n = 2) \qquad \text{or} \qquad k = 120 \quad (n = 4),$$

and by [8, p. 128], $\omega_{2n} = 2\gamma_{2n} + \lambda_{2n}$. Also, since n is even, $p_*(\theta_{4n-1}) = \omega_{2n}$. Suppose $\theta_{4n-1} = r\theta' + \alpha$, where θ' and α are defined as above, and suppose $p_*(\theta') = q\gamma_{2n} + s\lambda_{2n}$, $p_*(\alpha) = t\lambda_{2n}$. By Lemma 5.1, $t = 0$ or $t = k/2$; in either case, t is even. We have the relations

$$\omega_{2n} = 2\gamma_{2n} + \lambda_{2n} = p_*(\theta_{4n-1}) = p_*(r\theta' + \alpha) = rq\gamma_{2n} + (rs + t)\lambda_{2n},$$

and hence $rq = 2$. If $|r| = 2$, then $2s + t \equiv 1 \pmod k$, and since t is even, this is impossible; therefore $|r| = 1$, and again θ_{4n-1} is a generator.

PROPOSITION 5.3. *Let* $o^{2n+1}_{4n-1} \in \pi_{4n-1}(O(2n+1))$ *be a generator of infinite order, let* π: $O(2n+1) \to V_{2n+1,2}$ *be the usual bundle projection, and let* $\theta_{4n-1} \in \pi_{4n-1}(V_{2n+1,2})$ *be a generator selected as in* Lemma 5.2. *Then*

$$\pi_*(o^{2n+1}_{4n-1}) = \pm M_n \theta_{4n-1} + \beta,$$

where β *is of finite order and* $M_1 = 1$, $M_2 = 6$, $M_4 = 7!/4$, *and* $M_n = (2n-1)!/8$ *for* $n \neq 1, 2, 4$.

Proof. We use the commutative diagram

$$\begin{array}{ccc} \pi_{4n-1}(O(2n+1)) & \xrightarrow{\phi_*} & \pi_{4n-1}(U(2n)) \\ \downarrow \pi_* & & \downarrow \pi'_* \\ \pi_{4n-1}(V_{2n+1,2}) & \xrightarrow{\phi'_*} & \pi_{4n-1}(S_{4n-1}), \end{array}$$

together with the calculations in the proof of Theorem 4.2. In particular,

$$\pm 2^{\beta(n,2n+1)} a_n (2n-1)!\, \iota_{4n-1} = \pi'_* \phi_* (o^{2n+1}_{4n-1}) = \phi'_* \pi_* (o^{2n+1}_{4n-1})$$

$$= \phi'_*(M_n \theta_{4n-1} + \beta) = \pm 8 a_n M_n \iota_{4n-1},$$

for $n \geq 2$. If $n = 1$, a similar calculation shows that $+4\iota_3 = +4M_1\iota_3$. Since ι_{4n-1} is a generator of $\pi_{4n-1}(S_{4n-1}) = Z$, we can solve these equations for M_n.

THEOREM 5.4. *Let* $\partial\iota_{2n} \in \pi_{2n-1}(O(2n))$ *be the characteristic element of the bundle* $\pi'': O(2n+1) \to S_{2n}$. *Then the Samelson product*

$$\langle \partial\iota_{2n}, \partial\iota_{2n} \rangle \in \pi_{4n-2}(O(2n))$$

is of order either $a_n M_n$ *or* $2a_n M_n$.

Proof. The map π'' factors as $\pi'': O(2n+1) \xrightarrow{\pi} V_{2n+1,2} \xrightarrow{p} S_{2n}$; therefore

$$\pi_*(o^{2n+1}_{4n-1}) = \pm a_n M_n \omega_{2n} + \iota_{2n} \circ E\beta,$$

where β is of order at most 2, by Lemma 5.1. But from [2] we know that $\langle \partial\iota_{2n}, \partial\iota_{2n} \rangle = \pm\partial\omega_{2n}$. Thus

$$0 = \partial\pi_*(o^{2n+1}_{4n-1}) = \pm a_n M_n \partial\omega_{2n} + \partial(\iota_{2n} \circ E\beta) = \pm\langle \partial\iota_{2n}, \partial\iota_{2n} \rangle + (\partial\iota_{2n}) \circ \beta.$$

Thus if $\beta = 0$, $\langle \partial\iota_{2n}, \partial\iota_{2n} \rangle$ has order $a_n M_n$, and otherwise it has order $2a_n M_n$.

We remark that by entirely different methods Mahowald has shown that for $n \neq 1, 2, 4$, the order of $\langle \partial\iota_{2n}, \partial\iota_{2n} \rangle$ is $a_n M_n$. Using the detailed knowledge of the necessary homotopy groups in low dimensions, one can prove that the order is $a_n M_n$ for $n = 1, 3$, and that it is $2a_n M_n$ for $n = 2, 4$.

COROLLARY 5.5. *The Samelson product* $\langle \partial\iota_{2n}, \partial\iota_{2n} \rangle$ *is of even order for* $n \geq 2$.

These last results generalize Theorems 9.1 and 9.2 of James [7].

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