UNIVERSAL HOMOGENEOUS BOOLEAN ALGEBRAS

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A Boolean algebra $\mathfrak{B}$ of power α is said to be *universal* if every Boolean algebra of power at most α is isomorphically embeddable in $\mathfrak{B}$. $\mathfrak{B}$ is said to be *homogeneous* if for every subalgebra $\mathfrak{A}$ of $\mathfrak{B}$ of power less than α, every isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ can be extended to an automorphism of $\mathfrak{B}$. These notions are special cases of the general notions, due to Jónsson [4] and [5], of a K universal (relational) system and a K homogeneous system, where K is a class of systems. It follows from the general results of Jónsson and from known properties of Boolean algebras that if $2^{\alpha} = \alpha^{+}$ (the first cardinal greater than α), then there is, up to isomorphism, a unique universal homogeneous Boolean algebra of power α^{+}. However, Jónsson's results do not tell us which Boolean algebra the universal homogeneous one is. In this note we shall prove two theorems that identify the universal homogeneous Boolean algebra of power ω_1 as a very familiar Boolean algebra.

We let $S(\omega)$ denote the Boolean algebra of all subsets of the set ω of natural numbers, $S_{\omega}(\omega)$ the ideal of all finite subsets of ω, and $S(\omega)/S_{\omega}(\omega)$ the quotient algebra.

THEOREM 1. *Assume the continuum hypothesis* $2^{\omega} = \omega_1$. *Then* $S(\omega)/S_{\omega}(\omega)$ *is a universal homogeneous Boolean algebra of power* ω_1.

Instead of proving Theorem 1 as it stands, we shall prove a more general result. For this we need some more notation. Let $\mathfrak{A} = \langle A, +, \cdot, - \rangle$ be an arbitrary Boolean algebra. The direct power $\mathfrak{A}^{\omega}$, whose set of elements is the set A^{ω} of all functions on ω into A, is also a Boolean algebra. We denote by $\mathfrak{A}^*$ the quotient algebra of $\mathfrak{A}^{\omega}$ modulo the ideal of all functions $f \in A^{\omega}$ such that $f(n) = 0$ for all but finitely many n. For each $g \in A^{\omega}$, we let g^* be the element of A^* that g represents. The algebra $\mathfrak{A}^*$ is precisely the reduced power of $\mathfrak{A}$ modulo the filter of all cofinite subsets of ω (see [2]). It is obvious that if $\mathfrak{A}$ is the two-element Boolean algebra, then $\mathfrak{A}^{\omega}$ is isomorphic to $S(\omega)$, and $\mathfrak{A}^*$ is isomorphic to $S(\omega)/S_{\omega}(\omega)$.

THEOREM 2. *Assume the continuum hypothesis. Let* $\mathfrak{A}$ *be any Boolean algebra of power at most* ω_1. *Then* $\mathfrak{A}^*$ *is a universal homogeneous Boolean algebra of power* ω_1. *Moreover, if* $\mathfrak{B}$ *is any other Boolean algebra of power at most* ω_1, *then* $\mathfrak{A}^*$ *and* $\mathfrak{B}^*$ *are isomorphic.*

The general uniqueness theorem of Jónsson [5] shows that any two universal homogeneous Boolean algebras of the same power are isomorphic, so that the $\mathfrak{A}^*$ and $\mathfrak{B}^*$ above are isomorphic. The only step in the proof of Theorem 2 for which we shall need the continuum hypothesis is the proof that $\mathfrak{A}^*$ has power ω_1. Except for this point, the theorem is a consequence of the following lemma, which does not depend on the continuum hypothesis. For a Boolean algebra $\mathfrak{B} = \langle B, +, \cdot, - \rangle$, and a subset $C \subset B$, let $\mathfrak{B} \mid C$ denote the subalgebra of $\mathfrak{B}$ generated by C.

LEMMA 3. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be Boolean algebras, let* C *be a subset of* B *of power at most* ω, *and let* b *be an element of* B. *Then any isomorphism* f *of* $\mathfrak{B} \mid C$ *into* $\mathfrak{A}^*$ *can be extended to an isomorphism* f' *of* $\mathfrak{B} \mid C \cup \{b\}$ *into* $\mathfrak{A}^*$.

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By means of transfinite induction, it is easy to prove Theorem 2 from the lemma. Alternatively, Theorem 2 follows at once from the lemma and Theorem 2.5 in Morley and Vaught [6]. We now turn to the proof of the lemma.

We may assume that the set C is closed under the Boolean operations and is non-empty, so that C is the set of all elements of $\mathfrak{B} \mid C$. We may also assume that $b \notin C$. For each $c \in C$, choose $g(c) \in A^{\omega}$ such that $f(c) = g(c)^*$. List the elements of C, say $C = \{c_n: n < \omega\}$, in such a way that each $c \in C$ occurs infinitely many times in the list. We shall construct an infinite strictly increasing sequence of natural numbers

$$m(0) < m(1) < \cdots$$

and an element $g(b)$ of A^{ω} such that the following four conditions are satisfied for all n.

(1) If $c_n \leq b$, then $g(c_n)_p \leq g(b)_p$ whenever $m(2n) \leq p$.

(2) If not $c_n \leq b$, then not $g(c_n)_p \leq g(b)_p$, where $p = m(2n)$.

(3) If $b \leq c_n$, then $g(b)_p \leq g(c_n)_p$ whenever $m(2n+1) \leq p$.

(4) If not $b \leq c_n$, then not $g(b)_p \leq g(c_n)_p$, where $p = m(2n+1)$.

Once we have constructed the sequence m and the element g(b), we define $f'(b) = g(b)^*$. Remembering that each $c \in C$ occurs infinitely many times in the list c_n $(n < \omega)$, we see from (1) to (4) that for each $c \in C$, $f(c) \leq f'(b)$ if and only if $c \leq b$, and also $f'(b) \leq f(c)$ if and only if $b \leq c$. Each element of $\mathfrak{B} \mid C \cup \{b\}$ is of the form $b \cdot c + (-b) \cdot d$, where $c, d \in C$. We conclude that f can be extended (in a unique way) to an isomorphism f' of $\mathfrak{B} \mid C \cup \{b\}$ into $\mathfrak{A}^*$ with $f'(b) = g(b)^*$.

The sequence m and the element g(b) are built up by induction. Let $r < \omega$, and suppose that we have $m(0) < \cdots < m(2r-1)$ and an element $g(b, r)$ of A^{ω} such that (1) to (4) hold for all $n < r$ with $g(b, r)$ in place of g(b). For $r = 0$, we let $g(b, 0)$ be an arbitrary element of A^{ω}. We shall find $m(2r)$, $m(2r+1)$, and $g(b, r+1)$ such that $m(0) < \cdots < m(2r+1)$, such that (1) to (4) hold for all $n \leq r$ with $g(b, r+1)$ in place of g(b), and such that

(5) $$g(b, r+1)_p = g(b, r)_p \quad \text{whenever } p < m(2r).$$

There are three cases:

I. $c_r \leq b$ and not $b \leq c_r$.

II. Not $c_r \leq b$ and $b \leq c_r$.

III. Not $c_r \leq b$ and not $b \leq c_r$.

The case $b = c_r$ cannot arise, because $b \notin C$. Let $x \in A^{\omega}$ be the sum in the algebra $\mathfrak{A}^{\omega}$ of all $g(c_n)$ such that $n < r$ and $c_n \leq b$, with the convention that the empty sum is the zero of A^{ω}. Let $y \in A^{\omega}$ be the product in A^{ω} of all $g(c_n)$ such that $n < r$ and $b \leq c_n$, where the empty product is the unit of A^{ω}. Note that $x_p \leq y_p$ whenever $m(2r-1) \leq p$.

Case I. In this case $f(c_r) \leq y^*$ and not $y^* \leq f(c_r)$. Choose $m(2r)$ such that $m(2r-1) < m(2r)$ and $g(c_r)_p \leq y_p$ whenever $m(2r) \leq p$. Now choose $m(2r+1) > m(2r)$ such that not $y_p \leq g(c_r)_p$, where $p = m(2r+1)$. Define

$$g(b, r+1)_p = \begin{cases} g(b, r)_p & \text{if } p < m(2r), \\ y_p & \text{if } p \geq m(2r). \end{cases}$$

Case II. We have not $f(c_r) \leq x^*$, but $x^* \leq f(c_r)$. Choose $m(2r) > m(2r - 1)$ such that not $g(c_r)_p \leq x_p$, where $p = m(2r)$. Choose $m(2r + 1) > m(2r)$ so that $x_p \leq g(c_r)_p$ whenever $m(2r + 1) \leq p$. Now define

$$g(b, r+1)_p = \begin{cases} g(b, r)_p & \text{if } p < m(2r), \\ x_p & \text{if } p \geq m(2r). \end{cases}$$

Case III. This time, not $f(c_r) \leq x^*$ and not $y^* \leq f(c_r)$. Take

$$m(2r + 1) > m(2r) > m(2r - 1)$$

such that not $g(c_r)_p \leq x_p$ where $p = m(2r)$, and not $y_q \leq g(c_r)_q$ where $q = m(2r + 1)$. Let

$$g(b, r+1)_p = \begin{cases} x_p & \text{if } p = m(2r), \\ y_p & \text{if } p = m(2r+1), \\ g(b, r)_p & \text{otherwise.} \end{cases}$$

It is easy to verify that in each case $m(2r)$, $m(2r + 1)$, and $g(b, r + 1)$ have all the required properties. By (5), $g(b, r)_p = g(b, s)_p$ whenever $p < m(2r)$ and $p < m(2s)$. We define $g(b)_p = g(b, r)_p$, where r is such that $p < m(2r)$. It follows that (1) to (4) hold for $g(b)$, and our proof is complete.

Let α be an uncountable cardinal. It is curious that neither of the Boolean algebras $S(\alpha)/S_\omega(\alpha)$ and $S(\alpha)/I$ is homogeneous; $S(\alpha)$ is the algebra of all subsets of α, $S_\omega(\alpha)$ is the ideal of all finite subsets of α, and I is the ideal of all subsets of α of power less than α. $S(\alpha)/S_\omega(\alpha)$ fails to be homogeneous (if we assume $2^\omega < 2^\alpha$) because the subalgebra $\{0, \omega, \alpha - \omega, \alpha\}/S_\omega(\alpha)$ has an obvious automorphism that cannot be extended to the whole algebra. If α is not cofinal with ω, then $S(\alpha)/I$ fails to be homogeneous, because it has a properly increasing countable sequence $b_0 < b_1 < \cdots$ of elements whose supremum is the unit element, and another whose supremum is not the unit element. On the other hand, if α is cofinal with ω, then $S(\alpha)/I$ fails to be homogeneous, because it has a properly increasing sequence of type ω_1 of elements whose supremum is the unit element, and another whose supremum is not.

We conclude with some historical remarks. Our results in this note are closely related to some early results of Hausdorff. If X, Y are subsets of a Boolean algebra, then $X < Y$ means that $x < y$ for all $x \in X$, $y \in Y$. Hausdorff [3] proved (without the continuum hypothesis) that if X and Y are finite or countable simply ordered subsets of the algebra $S(\omega)/S_\omega(\omega)$ and $X < Y$, then there is an element z of the algebra such that $X < \{z\} < Y$. Lemma 3 is an improvement on Hausdorff's result, in the sense that Hausdorff's result is an easy consequence of Lemma 3. On the other hand, Hausdorff's result makes it very natural to guess that $S(\omega)/S_\omega(\omega)$ is universal and homogeneous. The author began thinking about the algebra $S(\omega)/S_\omega(\omega)$ after a conversation with Tarski on the above result of Hausdorff.

The existence and uniqueness of universal homogeneous Boolean algebras of each power α^+, where $\alpha^+ = 2^\alpha$, has been known ever since Jónsson introduced the notions. In the papers [4] and [5], Jónsson gave a very general sufficient condition for a class K of relational systems to contain a unique universal homogeneous system of power α^+, where $\alpha^+ = 2^\alpha$. He gave several examples of classes K for which his sufficient condition holds, but he did not mention Boolean algebras. However, Eva Kallin observed in 1957 that the class of all Boolean algebras satisfies Jónsson's condition, and she wrote Jónsson a letter containing a simple proof. (The author is indebted to Jónsson for looking up the letter and providing this information.) Several people appear to have noticed independently that the class of all Boolean algebras satisfies Jónsson's condition. A proof can be found in [1].

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