A UNIVERSAL MODEL FOR ERGODIC TRANSFORMATIONS ON SEPARABLE MEASURE SPACES

N. Dinculeanu and C. Foiaş

1. INTRODUCTION

Let $\mathscr{C}$ be the class of all objects (X, Σ, μ, T), where (X, Σ, μ) is a measure space with $X \in \Sigma$ and $\mu(X) = 1$, and where $T: X \to X$ is a μ-measurable measure-preserving transformation, that is, where for every $A \in \Sigma$ we have the relations

$$T^{-1}A \in \Sigma \quad \text{and} \quad \mu(T^{-1}A) = \mu(A).$$

Where no confusion can arise, we shall write (X, μ, T) instead of (X, Σ, μ, T).

If $(X, \mu, T) \in \mathscr{C}$, we denote by U_T the linear isometry of $L^2(X, \mu)$ into itself defined by

$$U_T f = f \circ T \quad \text{for } f \in L^2(X, \mu).$$

We say that $(X, \mu, T) \in \mathscr{C}$ is the *homomorphic image* of $(X_0, \mu_0, T_0) \in \mathscr{C}$ if there exists a linear isometry

$$\Phi: L^2(X, \mu) \to L^2(X_0, \mu_0)$$

satisfying the two conditions

1) $\Phi U_T = U_{T_0}\Phi$,

2) $\Phi(fg) = \Phi(f)\cdot\Phi(g) \quad$ for $f, g \in L^\infty(X, \mu)$

[the conditions imply that $\Phi L^\infty(X, \mu) \subset L^\infty(X_0, \mu_0)$]. For example, we can obtain such an isometry Φ from a measure-preserving transformation $\phi: X_0 \to X$ such that $\phi \circ T_0 = T \circ \phi$ by putting

$$\Phi f = f \circ \phi \quad \text{for } f \in L^2(X, \mu).$$

Under certain conditions, every such isometry can be obtained in this way (see [2, pp. 42-45] and [3, pp. 294-302]).

We say that $(X_0, \mu_0, T_0) \in \mathscr{C}$ is a *universal model* for a class $\mathscr{C}' \subset \mathscr{C}$ if every $(X, \mu, T) \in \mathscr{C}'$ is a homomorphic image of (X_0, μ_0, T_0).

In this paper we shall construct a universal model $(\hat{N}, \hat{\mu}, \hat{\tau}) \in \mathscr{C}$ for the class $\mathscr{C}_{e,s}$ of the (X, Σ, μ, T) such that T is *ergodic* and (X, μ) is separable (that is, there exists a countable subset $\Sigma' \subset \Sigma$ such that to every $A \in \Sigma$ and every $\varepsilon > 0$ there corresponds $A' \in \Sigma'$ with $\mu(A \triangle A') < \varepsilon$; or, equivalently, $L^2(X, \mu)$ is separable). The transformation $\hat{\tau}$ is invertible (modulo μ), but it is not ergodic, and the measure space $(\hat{N}, \hat{\mu})$ is not separable; in fact, a universal model for the class $\mathscr{C}_{e,s}$ cannot be separable.

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The idea of this model was suggested by an "heuristic proof" indicated in [2] (see the end of the paragraph "Comments on the ergodic theorem").

The following problems are still open:

I. Is there a universal *ergodic* model for $\mathscr{C}_{e,s}$?

II. Is $(\hat{N}, \hat{\mu}, \hat{\tau})$ a universal model for the class $\mathscr{C}_s$ of the separable but not necessarily ergodic measure spaces (X, μ, T)? If the answer is negative, is there a universal model for $\mathscr{C}_s$?

The main result is established in Section 5; the other paragraphs are introductory, and the proofs are given for the sake of the completeness.

2. THE SPACE $\hat{N}$

Consider the space $N = \{1, 2, \cdots\}$ equipped with the discrete topology, and let $\hat{N}$ be the Stone-Čech compactification of N. Then N is dense in $\hat{N}$, and every bounded complex sequence can be uniquely extended to a continuous function on $\hat{N}$ [1, p. 872], so that we can identify the algebra m of bounded complex sequences (with the usual multiplication of coordinates) with the algebra $C(\hat{N})$ of the complex continuous functions on $\hat{N}$, and $\hat{N}$ itself with the space of the maximal ideals of m.

Every point of N is open in $\hat{N}$, therefore the set $N_\infty = \hat{N} - N$ is compact.

LEMMA 1. *A point* $x \in \hat{N}$ *belongs to* N_∞ *if and only if*

$$|f(x)| \leq \limsup_{n \to \infty} |f(n)| \quad \textit{for every } f \in C(\hat{N}).$$

Suppose that $x = p \in N$, and consider the function $f \in C(\hat{N})$ defined by $f(n) = 1/n$ for $n \in N$. Then $\limsup_{n\to\infty} |f(n)| = 0$, but $|f(x)| = 1/p > 0$.

Conversely, let $x \in N_\infty$ and $f \in C(\hat{N})$. There exists a sequence $\{n_p\}$ in N such that $\lim_{p\to\infty} f(n_p) = f(x)$. In fact, we can suppose that $f(x) = 0$. If the desired sequence $\{n_p\}$ does not exist, then $\liminf |f(x)| > 0$. Put

$$g(n) = \begin{cases} 1/f(n) & \text{if } f(n) \neq 0, \\ 1 & \text{if } f(n) = 0. \end{cases}$$

Then $\{g(n)\} \in m$ and $g(n)f(n) = 1$, except for finitely many values n. It is now evident that $f(z)g(z) = 1$ on N_∞, and in particular $f(x) \neq 0$, which contradicts our hypothesis.

3. THE TRANSFORMATION $\hat{\tau}$

Consider the mapping $\tau: C(\hat{N}) \to C(\hat{N})$ defined by

$$(\tau f)(n) = f(n+1) \quad \text{for all } f \in C(\hat{N}) \text{ and } n \in N.$$

We remark that every function $g \in C(\hat{N})$ is of the form $g = \tau f$ for some $f \in C(\hat{N})$.

For every $x \in \hat{N}$, the mapping $f \to (\tau f)(x)$ is a continuous homomorphism of $C(\hat{N})$ onto the algebra of complex numbers; therefore there exists a point $\hat{\tau}x \in \hat{N}$ such that

$$(\tau f)(x) = f(\hat{\tau}x) \qquad \text{for every } f \in C(\hat{N}).$$

PROPOSITION 1. *The mapping* $\hat{\tau}: \hat{N} \to \hat{N}$ *is one-to-one and continuous, and* $\hat{\tau}(N_\infty) = N_\infty$.

Let $x_1, x_2 \in \hat{N}$ be such that $\hat{\tau}x_1 = \hat{\tau}x_2$. Let $g = \tau f$ be an arbitrary function of $C(\hat{N})$. Then $(\tau f)(x_1) = (\tau f)(x_2)$, that is, $g(x_1) = g(x_2)$. Therefore $x_1 = x_2$, whence $\hat{\tau}$ is one-to-one.

To prove that $\hat{\tau}$ is continuous, let $x_0 \in \hat{N}$, and let V be a neighbourhood of $\hat{\tau}x_0$ to the form

$$V = \{x: |f_j(x) - f_j(\hat{\tau}x_0)| < \varepsilon;\ f_j \in C(\hat{N}),\ j = 1, 2, \cdots, k\}.$$

Then

$$W = \{x: |(\tau f_j)(x) - \tau f_j(x_0)| < \varepsilon;\ j = 1, 2, \cdots, k\}$$

is a neighbourhood of x_0, and $\hat{\tau}W \subseteq V$.

We remark that if $n \in N$, then $\hat{\tau}n = n + 1$. If $x \in N_\infty$, then, by Lemma 1,

$$|f(\hat{\tau}x)| = |\tau f(x)| \leq \limsup_{n\to\infty} |(\tau f)(n)| = \limsup_{n\to\infty} |f(n+1)| = \limsup_{n\to\infty} |f(n)|$$

for every $f \in C(\hat{N})$. Therefore, again by Lemma 1, $\hat{\tau}x \in N_\infty$. It follows that $\hat{\tau}(N_\infty) \subseteq N_\infty$.

Conversely, let $y \in N_\infty$. For every function $f \in C(\hat{N})$, define the function $f_\tau \in C(\hat{N})$ by $f_\tau(1) = 0$ and $f_\tau(n+1) = f(n)$ for $n \in N$. It follows that $\tau(f_\tau) = f$. Moreover, $(\tau f)_\tau(z) = f(z)$ for $z \in N_\infty$. Since the mapping $f \to f_\tau(y)$ is a continuous homomorphism of $C(\hat{N})$ onto the algebra of the complex numbers, there exists a point $x \in \hat{N}$ such that

$$f(x) = f_\tau(y) \qquad \text{for every } f \in C(\hat{N}).$$

By virtue of Lemma 1, we conclude that $x \in N_\infty$. Consequently,

$$f(\hat{\tau}x) = (\tau f)(x) = (\tau f)_\tau(y) = f(y) \text{ for every } f \in C(\hat{N}),$$

and therefore $\hat{\tau}x = y$.

COROLLARY. *The restriction* τ_∞ *of* $\hat{\tau}$ *to* N_∞ *is a homeomorphism of* N_∞.

In fact, N_∞ is compact, τ_∞ is continuous and one-to-one, and $\tau_\infty(N_\infty) = N_\infty$.

Remark. There exists no fixed point of N_∞ for τ_∞. Indeed, suppose that $\tau_\infty y = y$ for some $y \in N_\infty$. Then

$$f(y) = (\tau f)(y) \qquad \text{for every } f \in C(\hat{N}),$$

and in particular, this is true for the function g defined by the sequence $\{0, 1, 0, 1, \cdots\}$. The function τg is defined by the sequence $\{1, 0, 1, 0, \cdots\}$, and since $g + \tau g = 1$ and $g \cdot \tau g = 0$, it follows that $2g(y) = 1$ and $g(y)^2 = 0$, which is impossible.

4. THE MEASURE $\hat{\mu}$

For every function $f \in C(\hat{N})$, put

$$|||f||| = \limsup \frac{|f(1) + f(2) + \cdots + f(n)|}{n}.$$

Then $|||f||| \leq \|f\|$, where

$$\|f\| = \sup_{x \in \hat{N}} |f(x)| = \sup_{n \in N} |f(n)|,$$

and $|||f|||$ is a seminorm on $C(\hat{N})$.

Consider now the set C_0 of functions $f \in C(\hat{N})$ such that the sequence

$$\left\{ \frac{f(1) + f(2) + \cdots + f(n)}{n} \right\}$$

converges. Then C_0 is a linear subspace of $C(\hat{N})$ containing all constant functions, and

$$I(f) = \lim_{n \to \infty} \frac{f(1) + f(2) + \cdots + f(n)}{n}$$

is a linear functional on C satisfying the inequality

$$|I(f)| \leq |||f||| \quad \text{for } f \in C_0.$$

Moreover, if $f \in C_0$ and $f \geq 0$, then $I(f) = |||f||| \geq 0$, and if $\lim_{n \to \infty} f(n) = c$, then $I(f) = c$; in particular, $I(1) = 1$.

PROPOSITION 2. *There exists a positive, regular Borel measure* $\hat{\mu}$ *on* $\hat{N}$ *such that* $(\hat{N}, \hat{\mu}, \hat{\tau}) \in \mathscr{C}$ *and*

$$\int f\, d\hat{\mu} = \lim_{n \to \infty} \frac{f(1) + f(2) + \cdots + f(n)}{n} \quad \text{for } f \in C_0.$$

Moreover, $\hat{\tau}$ *is invertible* (modulo $\hat{\mu}$), *but is not ergodic with respect to any measure* $\hat{\mu}$ *satisfying the preceding conditions.*

Using the Hahn-Banach theorem, we can extend I to a linear functional $\hat{I}$ on $C(\hat{N})$ such that

$$|\hat{I}(f)| \leq |||f||| \quad \text{for } f \in C(\hat{N}).$$

The functional $\hat{I}$ is positive. Indeed, let $f \geq 0$ be a function of $C(\hat{N})$. To prove that $\hat{I}(f) \geq 0$, we take a function $g \in C_0$ such that $f \leq g$ (for example, $g(x) \equiv \|f\|$). Then $\hat{I}(g) = |||g|||$ and $0 \leq g - f \leq g$, and therefore $|||g - f||| \leq |||g|||$. It follows that

$$\hat{I}(g) - \hat{I}(f) = \hat{I}(g - f) \leq |\hat{I}(g - f)| \leq |||g - f||| \leq |||g|||,$$

whence $\hat{I}(f) \geq \hat{I}(g) - |||g||| = 0$.

By the Riesz-Kakutani theorem, there exists a positive regular Borel measure $\hat{\mu}$ on $\hat{N}$ such that

$$\hat{I}(f) = \int f\,d\hat{\mu} \qquad \text{for } f \in C(\hat{N}).$$

In particular, for $f \in C_0$,

$$\int f\,d\hat{\mu} = I(f) = \lim_{n \to \infty} \frac{f(1) + \cdots + f(n)}{n}$$

and $\hat{\mu}(N) = I(1) = 1$.

Since $\hat{\tau}$ is continuous, it is $\hat{\mu}$-measurable. The transformation $\hat{\tau}$ is also measure-preserving. In fact, for every function $f \in C(\hat{N})$, we can write

$$\frac{(f - \tau f)(1) + (f - \tau f)(2) + \cdots + (f - \tau f)(n)}{n} = \frac{f(1) - f(n+1)}{n} \to 0.$$

Therefore $f - \tau f \in C_0$ and $I(f - \tau f) = 0$, and consequently $\hat{I}(f) = \hat{I}(\tau f)$. It follows that $\hat{\mu}(A) = \hat{\mu}(\hat{\tau}^{-1} A)$ for every $\hat{\mu}$-measurable set $A \subset \hat{N}$.

It remains to prove that $\hat{\tau}$ is not ergodic with respect to $\hat{\mu}$. Consider the function $f \in C(\hat{N})$ defined by

$$f(n) = \begin{cases} 1 & \text{if } (2k-1)^2 < n \leq (2k)^2, \\ 0 & \text{if } (2k)^2 < n \leq (2k+1)^2. \end{cases}$$

This function takes only the values 0 and 1; hence it is the characteristic function of a set $M \subset \hat{N}$ that is closed and open.

For every $n \in N$, let $h = h(n) \in N$ be defined by the inequalities

$$(2h-1)^2 < n \leq (2h+1)^2.$$

Then

$$\frac{\sum_{k=1}^{h-1} [(2k)^2 - (2k-1)^2]}{(2h+1)^2} \leq \frac{f(1) + \cdots + f(n)}{n} \leq \frac{\sum_{k=1}^{h} [(2k)^2 - (2k-1)^2]}{(2h-1)^2},$$

and a simple computation shows that

$$\frac{2h^2 - 3h + 1}{(2h+1)^2} \leq \frac{f(1) + \cdots + f(n)}{n} \leq \frac{2h^2 + h}{(2h-1)^2}.$$

Since $h \to \infty$ as $n \to \infty$, we deduce that

$$\lim_{n \to \infty} \frac{f(1) + \cdots + f(n)}{n} = \frac{1}{2};$$

therefore $f \in C_0$ and $\hat{\mu}(M) = \int f\,d\hat{\mu} = 1/2$.

On the other hand,

$$\frac{|\tau f(1) - f(1)| + |\tau f(2) - f(2)| + \cdots + |\tau f(n) - f(n)|}{n} \leq \frac{2h}{(2h-1)^2},$$

and therefore $I(|\tau f - f|) = 0$. It follows that

$$\hat{\mu}(M \triangle \hat{\tau}^{-1} M) = I(|\tau f - f|) = 0;$$

consequently $\hat{\tau}$ is not ergodic.

We shall show that $\hat{\mu}(N) = 0$, which implies that $\hat{\mu}$ is concentrated on N_∞. Indeed, every point $p \in N$ is open and closed in $\hat{N}$, and the characteristic function f_p of the set $\{p\}$ defined by $f_p(n) = \delta_{pn}$ is continuous. Since

$$\frac{f_p(1) + \cdots + f_p(n)}{n} = \frac{1}{n} \to 0 \qquad (p \leq n \to \infty),$$

$f_p \in \hat{C}_0$ and $\hat{\mu}(\{p\}) = I(f_p) = 0$. It follows that if we consider the transformation $\tau'\colon \hat{N} \to \hat{N}$ with $\tau'(n) = n$ for $n \in N$ and $\tau'(x) = \hat{\tau}(x)$ for $x \in N_\infty$, we obtain an invertible measurable transformation, $\hat{\mu}$-almost everywhere equal to $\hat{\tau}$; therefore $\hat{\tau}$ is itself invertible modulo $\hat{\mu}$.

Remark. The measure $\hat{\mu}$ is not unique. In the sequel, we assume that $\hat{\mu}$ is fixed.

5. THE MAIN RESULT

THEOREM. $(\hat{N}, \hat{\mu}, \hat{\tau})$ *is a universal model for the class* $\mathscr{C}_{e,s}$.

Let $(X, \mu, T) \in \mathscr{C}_{e,s}$. For every function $f \in L^1(\mu)$, the individual ergodic theorem for ergodic transformations [2, pp. 31-34] implies that

$$\lim_{n\to\infty} \frac{f(x) + f(Tx) + \cdots + f(T^{n-1}x)}{n} = \int_X f\, d\mu \tag{1}$$

μ-almost everywhere.

There exists a countable set $\mathscr{A} \subseteq L^\infty(X, \mu)$ with the following properties:

1) $f, g \in \mathscr{A}$ implies $fg \in \mathscr{A}$ and $\alpha f + \beta g \in \mathscr{A}$ for all rational complex numbers α, β; $f \equiv 1 \in \mathscr{A}$;

2) $f \in \mathscr{A}$ implies $\bar{f} \in \mathscr{A}$;

3) $\mathscr{A}$ is dense in $L^2(X, \mu)$;

4) $f \in \mathscr{A}$ implies $f \circ T \in \mathscr{A}$.

In fact, we can start from a countable set $\mathscr{A}_0$ dense in $L^2(X, \mu)$, then take the set $\mathscr{A}_1$ of functions of the form $f \circ T^n$ and $\bar{f} \circ T^n$ with $f \in \mathscr{A}_0$ and $n = 0, 1, 2, \cdots$, and finally take $\mathscr{A}$, the algebra over the complex rational numbers generated by $\mathscr{A}_1$. We may also suppose that

5) $|f(x)| \leq \|f\|_\infty$ for every $x \in X$ and $f \in \mathscr{A}$.

Since $\mathscr{A}$ is countable, there exists a μ-negligible set $N_0 \subset X$ such that (1) holds for every $x \in X - N_0$ and for every $f \in \mathscr{A}$. Let $y \in X - N_0$. For every $f \in \mathscr{A}$, the sequence $\{f(T^n y)\}$ is bounded, hence it belongs to m.

We now define the mapping $\Phi: \mathscr{A} \to C(\hat{N})$ by the equation

$$(\Phi f)(n) = f(T^n y) \quad \text{for } f \in \mathscr{A} \text{ and } n \in N. \tag{2}$$

From equation (1) we deduce that for $f \in \mathscr{A}$ the function Φf belongs to C_0 and that

$$\int \Phi f \, d\hat{\mu} = \int f \, d\mu . \tag{3}$$

We remark that Φf depends on the choice of the element y, but that the integral $\int \Phi f \, d\hat{\mu}$ is independent of y. In the sequel, y will be fixed.

It is easy to verify that Φ has the following properties:

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$ for $f, g \in \mathscr{A}$ and for all rational complex numbers α, β; $\Phi 1 = 1$;

(ii) $\Phi(fg) = \Phi f \cdot \Phi g$ for $f, g \in \mathscr{A}$;

(iii) $\Phi(\bar{f}) = \overline{\Phi f}$ for $f \in \mathscr{A}$;

(iv) $\Phi U_T = U_{\hat{\tau}} \Phi$.

For instance, for every $f \in \mathscr{A}$ and $n \in N$ we have the relations

$$\Phi U_T f(n) = \Phi(f \circ T)(n) = f(T^{n+1} y) = \Phi f(n+1) = \Phi f(\hat{\tau} n) = U_{\hat{\tau}} \Phi f(n).$$

From (3) we deduce that Φ is an isometry from $\mathscr{A} \subset L^2(X, \mu)$ into $C(\hat{N}) \subseteq L^2(\hat{N}, \hat{\mu})$:

$$(f, g) = \int f\bar{g} \, d\mu = \int \Phi(f\bar{g}) \, d\hat{\mu} = \int \Phi f \cdot \Phi(\bar{g}) \, d\hat{\mu} = \int \Phi f \cdot \overline{\Phi g} \, d\hat{\mu} = (\Phi f, \Phi g).$$

Since $\mathscr{A}$ is dense in $L^2(X, \mu)$, it follows that Φ can be extended to a linear isometry from $L^2(X, \mu)$ into $L^2(\hat{N}, \hat{\mu})$; we shall also denote the extension by Φ. Equation (iv) remains valid for the extended Φ.

It remains to prove that equation (ii) remains valid for $f, g \in L^\infty(X, \mu)$. We shall prove more, namely, that the equation remains valid for $f \in L^2(\mu)$ and $g \in L^\infty(\mu)$. We shall divide the proof into several parts.

a) *Equation* (ii) *is valid for* $f \in \mathscr{A}$ *and* $g \in L^\infty(\mu)$. In fact, there exists a sequence $\{g_n\}$ in $\mathscr{A}$ converging to g in $L^2(\mu)$. Since f is bounded, $fg_n \to fg$ in $L^2(\mu)$. Therefore $\Phi(g_n) \to \Phi(g)$ and $\Phi(fg_n) \to \Phi(fg)$ in $L^2(\mu)$. Since Φf is bounded (being continuous on $\hat{N}$) we see that

$$\Phi f \cdot \Phi g_n \to \Phi f \cdot \Phi g \quad \text{in } L^2(\mu).$$

For every n, $\Phi(fg_n) = \Phi(f)\Phi(g_n)$; therefore, passing to the limit, we obtain the relation

$$\Phi(fg) = \Phi f \cdot \Phi g .$$

b) *If* $g \in L^\infty(\mu)$, *then* $\Phi g \in L^\infty(\hat{\mu})$ *and*

$$\|\Phi g\|_\infty \leq \|g\|_\infty . \tag{4}$$

For $f \in \mathscr{A}$, condition 5) implies that

$$\max_{x \in \hat{N}} |\Phi f(x)| = \sup_n |f(T^n y)| \leq \|f\|_\infty,$$

so that (4) is valid for $g = f \in \mathscr{A}$. Hence, denoting by $\overline{\mathscr{A}}$ the closure of $\mathscr{A}$ in $L^\infty(\mu)$, we deduce from continuity that Φ is a linear multiplicative mapping of $\overline{\mathscr{A}}$ into $L^\infty(\hat{\mu})$ and that it satisfies (4). But $\overline{\mathscr{A}}$ is a subalgebra of $L^\infty(\mu)$ satisfying the conditions

α) $1 \in \overline{\mathscr{A}}$;

β) $f \in \overline{\mathscr{A}}$ implies $\bar{f} \in \overline{\mathscr{A}}$.

For $c > 0$, let ϕ_c be defined on the set of complex numbers by

$$\phi_c(z) = \begin{cases} z & \text{if } |z| < c, \\ c\dfrac{z}{|z|} & \text{if } |z| \geq c. \end{cases}$$

Evidently ϕ_c is continuous and can be uniformly approximated on $|z| \leq C$ by polynomials $p(z, \bar{z})$ in z and $\bar{z}$. Thus, if $f \in \overline{\mathscr{A}}$ and $\|f\|_\infty \leq C$, then $\phi_c \circ f$ is the limit in $L^\infty(\mu)$ of elements of the form $p(f, \bar{f}) \in \overline{\mathscr{A}}$; hence $\phi_c \circ f \in \overline{\mathscr{A}}$ for all $c > 0$ and all $C < \infty$.

Now let $g \in L^\infty(\mu)$ be given, and suppose $f_n \in \mathscr{A}$ and $f_n \to g$ in $L^2(\mu)$. Taking $\varepsilon > 0$, $c = \|g\|_\infty + \varepsilon$, and $h_n = \phi_c \circ f_n$, we deduce that $h_n \in \overline{\mathscr{A}}$ and $h_n \to g$ in $L^2(\mu)$; hence $\Phi h_n \to \Phi g$ in $L^2(\hat{\mu})$. Moreover, we may suppose that $\Phi h_n \to \Phi g$, $\hat{\mu}$-almost everywhere. Since (4) is valid for the functions of $\overline{\mathscr{A}}$, we see that

$$\|\Phi h_n\|_\infty \leq c = \|g\|_\infty + \varepsilon.$$

Thus, $|\Phi g| \leq \|g\|_\infty + \varepsilon$ $\hat{\mu}$-almost everywhere, that is, $\|\Phi g\|_\infty \leq \|g\|_\infty + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain (4) for every $g \in L^\infty(\mu)$.

c) *Equation* (ii) *holds for* $f \in L^2(\mu)$ *and* $g \in L^\infty(\mu)$. Let $\{f_n\}$ be a sequence in $\mathscr{A}$ such that $f_n \to f$ in $L^2(\mu)$. Since $g \in L^\infty(\mu)$, we see that $f_n g \to fg$. Therefore

$$\Phi(f_n) \to \Phi(f) \quad \text{and} \quad \Phi(f_n g) \to \Phi(fg) \quad \text{in } L^2(\hat{\mu}).$$

Since by b) $\Phi g \in L^\infty(\hat{\mu})$, we deduce that $\Phi(f_n)\Phi(g) \to \Phi(f)\Phi(g)$. For every n, a) implies that $\Phi(f_n g) = \Phi(f_n)\Phi(g)$; therefore, passing to the limit, we conclude that $\Phi(fg) = \Phi(f)\Phi(g)$.

Remark. The countability hypothesis was used only to get a point $y \in X$ such that (1) holds for every function of a class $\mathscr{A}$ satisfying conditions 1) to 4), and this last property was indeed used in the proof; therefore $(\hat{N}, \hat{\mu}, \hat{\tau})$ is a universal model for the class of all ergodic transformations possessing the preceeding property. An interesting problem is whether $(\hat{N}, \hat{\mu}, \hat{\tau})$ itself has this property.

PROPOSITION 3. *There exists no separable universal model for the class* $\mathscr{C}_{e,s}$.

Suppose that (X_0, μ_0, T_0) is a universal model for $\mathscr{C}_{e,s}$. Let X be the complex unit circle, and let μ denote Haar measure on X. Let $c \in X$ be such that $c^n \neq 1$ for $n = \pm 1, \pm 2, \cdots$, and define the transformation $T\colon X \to X$ by $Tx = cx$

for $x \in X$. Then $(X, \mu, T) \in \mathscr{C}_{e,s}$ [2, pp. 25-30]. Moreover, T is invertible. Consider the unitary operator U_T on $L^2(X, \mu)$ induced by T. The function $f(x) = x$ satisfies the condition

$$f(cx) = c\,f(x) \qquad \text{for } x \in X,$$

that is, $U_T f = cf$.

Let $\Phi\colon L^2(X, \mu) \to L^2(X_0, \hat{\mu}_0)$ be a linear isometry such that $\Phi U_T = U_{T_0}\Phi$. Then

$$U_{T_0}\Phi f = \Phi U_T f = \Phi c f = c \Phi f;$$

therefore c is a proper value for U_{T_0}. Since the set of numbers $c \in X$ with $c^n \neq 1$ for $n = \pm 1, \pm 2, \cdots$ is uncountable, it follows that the set of proper values of U_{T_0} is uncountable, and therefore $L^2(X_0, \mu_0)$ contains an uncountable family of orthogonal elements different from 0. We deduce that (X_0, μ_0) is not separable.

REFERENCES

1. N. Dunford and S. Schwarz, *Linear operators, Part II,* Interscience, New York, 1958.

2. P. Halmos, *Lectures on ergodic theory,* Chelsea, New York, 1956.

3. J. von Neumann, *Collected works,* vol. II, Pergamon Press, New York, 1961.

Mathematical Institute of the Rumanian Academy
Bucarest