MULTIPLICATIONS ON SO(3)

C. M. Naylor

1. INTRODUCTION

A multiplication on a space X (with base point $*$) will be defined to be a map $\mu\colon X \times X \to X$ such that $\mu(x, *) = \mu(*, x) = x$ for all x. Two multiplications μ_1 and μ_2 will be said to be homotopic if μ_1 is homotopic to μ_2 relative to $X \vee X$. The problem of enumerating the homotopy classes of multiplications that a given space may possess has been studied by James [3] for spheres, and by Arkowitz and Curjel [1] for finite CW-complexes. We shall prove the following theorem.

THEOREM 1.1. *There exist precisely* 768 *distinct homotopy classes of multiplications on* SO(3).

2. RESTATEMENT OF THE PROBLEM

SO(3) is homeomorphic to 3-dimensional real projective space P^3. We use K to denote the reduced product $P^3 \wedge P^3 = P^3 \times P^3/P^3 \vee P^3$. By [1], P^3 has as many multiplications as there are elements of $[K, P^3]$, the set of homotopy classes of base-point-preserving maps from K to P^3; since K is simply connected, the latter is clearly equivalent to $[K, S^3]$.

The space K has a standard CW-structure (see Section 5). If we write $K^{(n)}$ for the n-skeleton, then K is obtained from $K^{(5)}$ by attaching one 6-cell by means of a map of its boundary $S^5 \xrightarrow{h} K^{(5)}$. By [5], the following is an exact sequence of groups (we write Σ for suspension):

$$[S^5, S^3] \xleftarrow{h^*} [K^{(5)}, S^3] \leftarrow [K, S^3] \leftarrow [S^6, S^3] \xleftarrow{\Sigma h^*} [\Sigma K^{(5)}, S^3] \leftarrow \cdots .$$

Since $[S^6, S^3] \simeq \pi_6(S^3) \simeq Z_{12}$, Theorem 1.1 is a consequence of the following three propositions.

PROPOSITION 2.1. $h^* = 0$.

PROPOSITION 2.2. $\Sigma h^* = 0$.

PROPOSITION 2.3. $[K^{(5)}, S^3]$ has order 2^6.

3. PROOFS OF PROPOSITIONS 2.1 AND 2.2

Proposition 2.1 asserts that gh is null-homotopic for each $g\colon K^{(5)} \to S^3$. Denote $K/K^{(2)}$ by L, and the natural projection $K \to L$ by p. Then the following implies 2.1.

PROPOSITION 3.1. *Let* $\tilde{g}\colon L^{(5)} \to S^3$ *be any map. Then* $\tilde{g}ph \sim *$.

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Proposition 3.1 is proved by determining completely the structure of L as a CW-complex. This structure is described in Proposition 3.2, which we shall prove in Section 5.

PROPOSITION 3.2. a) $L^{(5)}$ *has the homotopy type of* $\Sigma^3 P^2 \vee A \vee \Sigma^3 P^2 = M^{(5)}$, *where* $A = (S_1^3 \vee S_2^3) \bigcup_a e^4$, *and where in turn* $a: S^3 \to S_1^3 \vee S_2^3$ *is of type* (2, 2).

b) *The homotopy equivalence of part* a) *extends to a homotopy equivalence* $L \sim M$, *where the attaching map for* e^6 *in* M *maps* S^5 *into* $S_1^3 \vee S_2^3$.

c) *We denote the attaching map for* e^6 *in* M *by* (ph)'. *Then* (ph)' *is a multiple of the universal Whitehead product in* $\pi_5(S_1^3 \vee S_2^3)$.

Assuming Proposition 3.2, we now prove Propositions 3.1 (thus also 2.1) and 2.2.

Proof of Proposition 3.1. Let $\tilde{g}: L^{(5)} \to S^3$ be a map. We denote the corresponding map from $M^{(5)}$ to S^3 by $\tilde{g}'$. By Proposition 3.2, $\tilde{g}'(ph)' = (\tilde{g}' \mid A)(ph)'$ is a Whitehead product in S^3, and therefore null-homotopic. This implies that $\tilde{g}ph$ is null-homotopic, which proves Proposition 3.1.

Proof of Proposition 2.2. As before, we must show that $g(\Sigma h) \sim *$ for each $g: \Sigma K^{(5)} \to S^3$. The 4-skeleton of $\Sigma K^{(5)}$ consists of an S^3 with two 4-cells attached, each by a map of degree ± 2. We may thus take g to be trivial on the 3-skeleton S^3. In this case there is a map $\tilde{g}: \Sigma L^{(5)} \to S^3$ such that

$$g(\Sigma h) = \tilde{g}(\Sigma p)(\Sigma h) = \tilde{g}(\Sigma ph) \sim * .$$

The last relation follows from the fact that the suspension of a Whitehead product is trivial. This proves Proposition 2.2.

4. $[K^{(5)}, S^3]$

To determine the order of $[K^{(5)}, S^3]$, we first show that this group is isomorphic to each of the cohomotopy groups $\pi^{n+3}(\Sigma^n K^{(5)})$ $(n \geq 1)$. When $n > 2$, the order of $\pi^{n+3}(\Sigma^n K^{(5)})$ can be computed to be 2^6 by means of the cohomotopy spectral sequence. For an explicit description of the filtration of π^{n+3} that one obtains, see [4, p. 116]. The differentials in E^2 of the spectral sequence are Steenrod operations, and they are easily computed. The only differential in E^3 that affects the computation is the Adem operation Φ, which is defined on a subgroup of

$$H^{n+2}(\Sigma^n K^{(5)}; Z).$$

Since $H^{n+2}(\Sigma^n K^{(5)}; Z)$ is zero, this presents no difficulties. Thus the following implies Proposition 2.3:

PROPOSITION 4.1. $[K^{(5)}, S^3] \simeq \pi^{n+3}(\Sigma^n K^{(5)})$ *for all* $n \geq 1$.

Proof. S^3 has as classifying space PQ^∞, infinite-dimensional quaternion projective space; thus S^3 is homotopy-equivalent to ΩPQ^∞. Therefore

$$[K^{(5)}, S^3] \simeq [\Sigma K^{(5)}, PQ^\infty].$$

As a cell complex, $PQ^\infty \sim S^4 \cup e^8 \cup \cdots$. Since $\Sigma K^{(5)}$ has dimension 6, we have the isomorphism $[K^{(5)}, S^3] \simeq [\Sigma K^{(5)}, S^4]$. We are now in the stable range:

$$[\Sigma K^{(5)}, S^4] \xrightarrow{\Sigma} [\Sigma^2 K^{(5)}, S^5] \xrightarrow{\Sigma} \cdots$$

are isomorphisms, and 4.1 is proved.

5. STRUCTURE OF THE COMPLEX $L \simeq (K/K^2)$

P^3 has the usual CW-decomposition $* = P^0 \subset P^1 \subset P^2 \subset P^3$; therefore $P^3 \times P^3$ has a CW-structure with cells $P^i \times P^j$. The homology of P^3 is determined by $dP^3 = 0$, $dP^2 = 2P^1$, $dP^1 = 0$; this gives an induced differential on $P^3 \times P^3$, in the usual fashion. The Z_2-cohomology ring of P^3 has a 1-dimensional generator u with $u^4 = 0$. One finds that $H^i(P^3 \times P^3; Z_2)$ is spanned by $u^i \times u^j$ $(j, i - j \leq 3)$, so that the Steenrod operations in $P^3 \times P^3$ can be computed from the Cartan formula. The following is obvious:

LEMMA 5.1. *Let* U *and* V $(U \subset V)$ *be subcomplexes of* $P^3 \times P^3$. *Let* $q\colon V \to V/U$ *be the natural projection. Then* $H^*(V; Z_2)$ *is a direct summand of* $H^*(P^3 \times P^3; Z_2)$, *and* $q^*\colon H^*(V/U; Z_2) \to H^*(V; Z_2)$ *is a monomorphism.*

We also recall the following well-known fact concerning spaces X and Y:

LEMMA 5.2. $\pi_n(X \vee Y) \simeq \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y)$, *where the third summand is embedded by the homotopy boundary operator.*

We now study the complex $L = K/K^{(2)}$. A cell in L will be denoted by $P^i \times P^j$ if it is the image of $P^i \times P^j$ under the projection $P^3 \times P^3 \to L$. Clearly, $L^{(3)}$ is homotopy-equivalent to $S^3_1 \vee S^3_2$. The attaching maps for $P^1 \times P^3$ and $P^3 \times P^1$ have local degree 0 and are thus trivial, since the homology differential on these cells is 0. By the same argument, we see that $P^2 \times P^2$ is attached by a map $a\colon S^3 \to S^3_1 \vee S^3_2$ of type (2, 2). If we denote $S^3_1 \vee S^3_2 \bigcup_a e^4$ by A, then $L^{(4)}$ is homotopy-equivalent to $S^4_1 \vee A \vee S^4_2$.

There are two 5-cells: $P^2 \times P^3$ and $P^3 \times P^2$. The cell $P^3 \times P^2$ is attached by a map $f\colon S^4 \to S^4_1 \vee A$. By Lemma 5.2, f has the form $f_1 + f_2$ $(f_1 \in \pi_4(S^4)$, $f_2 \in \pi_4(A))$. By an argument similar to the above, we find that $\deg f_1 = 2$; thus, to complete the proof of 3.2 a), it remains to show that $f_2 \sim *$, for then, by symmetry, $L^{(5)}$ is homotopy-equivalent to $\Sigma^3 P^2_1 \vee A \vee \Sigma^3 P^2_2$. To see that $f_2 \sim *$, consider

$$L^{(4)} \bigcup_f (P^3 \times P^2)/S^4_1 \vee S^4_2,$$

which is homotopy-equivalent to $A \bigcup_{f_2} e^5$. From the homotopy exact sequence of $(A, S^3_1 \vee S^3_2)$, we see that f_2 factors through $S^3_1 \vee S^3_2$. Let $\pi_i\colon S^3_1 \vee S^3_2 \to S^3_i$ denote the projection on the i-th factor (i = 1, 2). Denote by T the space

$$L^{(4)} \bigcup_f (P^3 \times P^2)/S^4_1 \vee S^3_2 \vee S^4_2,$$

which is homotopy-equivalent to $\Sigma^2 P^2 \bigcup_{\pi_1 f_2} e^5$. By Lemma 5.3, if $\pi_1 f_2$ is non-trivial, then there is $v \in H^3(T; Z_2)$ with $Sq^2 v \neq 0$. By Lemma 5.1, a similar statement would be true for $P^3 \times P^3$, a contradiction. Thus $\pi_i f_2 \sim *$ for i = 1, 2. By Lemma 5.2, $f_2 \sim *$, and this proves 3.2 a).

We now turn to a proof of 3.2 b) and c). $P^3 \times P^3$ is attached by ph, $S^5 \to L^{(5)}$. By Lemma 5.2, $ph = g_1 + g_2 + g$, with $g_1, g_2 \in \pi_5(\Sigma^3 P^2)$, $g \in \pi_5(A)$.

LEMMA 5.3. *Let* $f: S^{n+2} \to \Sigma^n P^2$. *Then* f *factors through* $(\Sigma^n P^2)^{(n+1)}$ $(= S^{n+1})$ *for* $n > 0$. *If* $T = \Sigma^n P^2 \cup_f e^{n+3}$ *and* v *is the generator of* $H^{n+1}(T, Z_2)$, *then for* $n > 1$, $f \sim *$ *if and only if* $Sq^2 v = 0$.

Proof. The first assertion follows from the homotopy exact sequence of $(\Sigma^n P^2, S^{n+1})$. Assume now that f factors through S^{n+1}, and denote $S^{n+1} \cup_f e^{n+3}$ by U. Let $i: U \to T$ be the injection. Then the second assertion follows from the fact that i^* is a Z_2-cohomology isomorphism in dimensions $n+1$ and $n+3$, and from well-known facts about U. Lemma 5.3 leads to the following well-known result.

COROLLARY 5.4. $\Sigma^n P^3 \simeq S^{n+3} \vee \Sigma^n P^2$ $(n \geq 2)$.

Lemmas 5.3 and 5.1 together now show that $g_1, g_2 \sim *$. It remains to show that g factors through $S_1^3 \vee S_2^3$ and is a Whitehead product. We recall that the latter is the same as saying $\pi_i g \sim *$ (i = 1, 2), where $\pi_i: S_1^3 \vee S_2^3 \to S_i^3$.

If g is not deformable into $S_1^3 \vee S_2^3$, then it determines a nonzero element of $\pi_5(A, S_1^3 \vee S_2^3)$. By [2, II, Theorem II],

$$q_*: \pi_5(A, S_1^3 \vee S_2^3) \to \pi_5(A/S_1^3 \vee S_2^3) \simeq \pi_5(S^4)$$

is an isomorphism, where q denotes the projection, and thus qg is nontrivial. This implies that for

$$L/\Sigma^3 P_1^2 \vee S_1^3 \vee S_2^3 \vee \Sigma^3 P_2^2 \quad \left(= S^4 \bigcup_{qg} e^6\right),$$

we have a $v \in H^4\left(S^4 \cup_{qg} e^6; Z_2\right)$ with $Sq^2 v \neq 0$. By 5.1, this is again a contradiction, and 3.2 b) is proved.

It remains to show that $\pi_i g \sim *$ (i = 1, 2). The following lemma is another application of the triad theorem.

LEMMA 5.5. *If* $g: S^{n+5} \to S_1^{n+3} \vee S_2^{n+3}$ *is nontrivial, then it is nontrivial as a map* $S^{n+5} \to \Sigma^n A$, *for* $n > 0$.

Using 5.5, we see that if $\pi_1 g$ or $\pi_2 g$ is nontrivial, then the top-dimensional cell of $\Sigma^n L$ has a nontrivial attaching map for $n > 0$. If we recall that

$$\Sigma(X \wedge Y) \simeq \Sigma X \wedge Y \simeq X \wedge \Sigma Y$$

(ΣX is $S^1 \wedge X \simeq X \wedge S^1$), we see that

$$\Sigma^4(P^3 \wedge P^3) \simeq \Sigma^2 P^3 \wedge \Sigma^2 P^3 \simeq S^{10} \vee \Sigma^7 P^2 \vee \Sigma^7 P^2 \vee (\Sigma^2 P^2 \wedge \Sigma^2 P^2),$$

by Corollary 5.4. Thus $\Sigma^4(P^3 \wedge P^3)$ (and thus $\Sigma^4 L$) has its top-dimensional cell attached by a trivial map. By the above remarks, this completes the proof of 3.2 c).

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Stanford University