LOCAL HOMEOMORPHISMS OF EUCLIDEAN SPACE ONTO ARBITRARY MANIFOLDS

Marshall M. Cohen

INTRODUCTION

We define an n-manifold as a separable, connected metric space in which each point has an open neighborhood homeomorphic to Euclidean n-space R^n. A local homeomorphism of a space X into a space Y is a mapping such that each point of X has an open neighborhood that goes homeomorphically onto an open set in Y. The main purpose of this paper is to point out (Theorem 1) that for each manifold M^n there exists a local homeomorphism f of R^n onto M^n.

Intuitively we think of M^n as a surface and of R^n as an infinitely long piece of tape, and we ask whether the piece of tape can be wrapped around the surface. Once we know that this can be done, it is natural to investigate the "most economical" wrapping of R^n onto M^n. That is, if f is a local homeomorphism as above, we let N(f) be the supremum of the cardinalities of the sets $f^{-1}(x)$, for $x \in M^n$. We define the *wrapping number:*

$$w(M^n) = \inf \{N(f) \mid f \text{ is a local homeomorphism of } M^n \text{ onto } R^n\} .$$

It is easy to see that $w(M^n)$ is countable, and that when M^n is compact, $w(M^n)$ is finite. We prove (Theorem 2) that if M^n is compact and $w(M^n) = 2$, then M^n has the n-sphere S^n as a twofold covering space. Theorem 3 asserts that if M^n is compact, is not the n-sphere, and can be covered with r open cells, then $w(M^n) \leq r$.

We remark that if M^n is a differentiable or piecewise linear manifold, then the local homeomorphisms constructed in Theorems 1 and 3 can be constructed so as to be differentiable or piecewise linear immersions, respectively.

Finally, Theorem 4 is an analogue of Theorem 1 for manifolds with nonempty connected boundary.

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1. CONSTRUCTION OF THE LOCAL HOMEOMORPHISM

We shall let $B^n(r)$ denote the closed ball of radius r about the origin in R^n. We write $B^n(1) = B^n$. $\mathring{B}^n$ denotes the interior of B^n.

If f is a mapping, then $|f|$ denotes the image of f.

If $A \subset X$, then Cl A and Int A are the closure and interior of A in X.

An n-cell Q in the manifold M^n is said to be *flat* if there exists an embedding f of $B^n(2)$ into M^n such that $Q = f(B^n(1))$. Any homeomorphism of a cell Q onto a

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flat n-cell Q' in M^n is called a *flat embedding* of Q into M^n. We assume a familiarity with the elementary properties of flat cells.

To avoid discussion of trivial cases we shall suppose, in all that follows, that our manifolds are of dimension $n > 1$. Theorems 1 to 4 are all true in dimension one.

LEMMA 0. *If* g *is a flat embedding of* $B^{n-1} \times [i, i+3]$ *into the manifold* M^n, Q *is a flat* n-*cell in* M^n *disjoint from* $|g|$, *and* U *is an open neighborhood of* Q, *then there exists a flat embedding* f *of* $B^{n-1} \times [i, i+3]$ *into* M^n *such that*

(i) $f(\mathring{B}^{n-1} \times (i+2, i+3)) \supset Q$,

(ii) $f(B^{n-1} \times [i+2, i+3]) \subset U$,

(iii) $f | B^{n-1} \times [i, i+1] = g \mid B^{n-1} \times [i, i+1]$.

We omit the proof.

The crux of the matter is now given by Lemma 1.

LEMMA 1. *If* M^n *is an* n-*dimensional manifold, then there exists a mapping* f *of* $B^{n-1} \times [0, \infty)$ *onto* M^n *such that*

1) $f(\mathring{B}^{n-1} \times (0, \infty) = M^n$,

2) $f \mid B^{n-1} \times [2i, 2i+3]$ *is a homeomorphism* $(i = 0, 1, 2, \cdots)$.

Proof. Let $Q_1, Q_2, Q_3, \cdots$ be an infinite sequence of flat cells (not necessarily distinct) in M^n, such that M^n is the union of the Q_i and $Q_i \cap Q_{i+1} = \emptyset$ for all i.

Since Q_1 is flat and $Q_1 \cap Q_2 = \emptyset$, there exists a flat embedding f_1 of $B^{n-1} \times [0, 3]$ into M^n such that

$$f_1(\mathring{B}^{n-1} \times (2, 3)) \supset Q_1 \quad \text{and} \quad f_1(B^{n-1} \times [2, 3]) \cap Q_2 = \emptyset.$$

Suppose that flat embeddings f_j $(1 \leq j \leq r)$ of $B^{n-1} \times [2j-2, 2j+1]$ into M^n have been defined so that

(i) $f_j(\mathring{B}^{n-1} \times (2j, 2j+1)) \supset Q_j$,

(ii) $f_j(B^{n-1} \times [2j, 2j+1]) \cap Q_{j+1} = \emptyset$,

(iii) $f_j \mid B^{n-1} \times [2j-2, 2j-1] = f_{j-1} \mid B^{n-1} \times [2j-2, 2j-1]$ if $j \geq 2$.

Since $f_r(B^{n-1} \times [2r, 2r+1])$ is a flat cell disjoint from Q_{r+1}, there exists an extension of $f_r | B^{n-1} \times [2r, 2r+1]$ to a flat embedding g_{r+1} of $B^{n-1} \times [2r, 2r+3]$ into M^n such that $|g_{r+1}| \cap Q_{r+1} = \emptyset$. Note that $M^n - Q_{r+2}$ is a neighborhood of Q_{r+1}, and apply Lemma 0; thus there exists a flat embedding f_{r+1} of $B^{n-1} \times [2r, 2r+3]$ into M^n such that (i) to (iii) hold, with $j = r+1$.

Proceeding inductively, we get an infinite sequence $f_1, f_2, \cdots$. Define $f: B^{n-1} \times [0, \infty) \to M^n$ by the condition that

$$f \mid B^{n-1} \times [2j-2, 2j+1] = f_j.$$

From (i) and (iii) and the fact that each f is a homeomorphism, it follows that f is a well-defined map with the desired properties. Q. E. D.

THEOREM 1. *If* M^n *is a topological* n-*manifold, then there exists a local homeomorphism of* R^n *onto* M^n.

Proof. Let g be a homeomorphism of R^n onto $\mathring{B}^{n-1} \times (0, \infty)$, and let f be as in Lemma 1. Then $h = fg$ is a local homeomorphism of R^n onto M^n.

2. THE CASE $w(M^n) = 2$

This section will be devoted to a proof of the following theorem.

THEOREM 2. *If* M^n *is a compact* n-*manifold and* $w(M^n) = 2$, *then* M^n *has* S^n *as a twofold covering space.*

Proof. By hypothesis there exists a local homeomorphism f of R^n onto M^n such that $f^{-1}f(x)$ consists of at most two points, for each x in R^n. Let X be the set of all points $x \in R^n$ such that $f^{-1}f(x) = x$. If $x \in R^n - X$, let T(x) denote the other point having the same image as x.

(1). X is compact. Indeed, since f is a local homeomorphism, $R^n - X$ is open. Thus X is closed. To see that X is bounded, note that by the compactness of M^n, there exists an integer r such that $f(B^n(r)) = M^n$. Since

$$f(X - B^n(r)) = f(X) - f(B^n(r)) = f(X) - M^n = \emptyset,$$

X is included in $B^n(r)$.

(2). T is a homeomorphism of $R^n - X$ onto itself. In fact, T is a local homeomorphism that is both one-to-one and onto.

(3). T takes sequences converging to infinity onto sequences converging to X, and vice versa. To see this, suppose $\{x_i\}$ is a sequence of points converging to infinity such that $\{T(x_i)\}$ does not converge to X. By taking a subsequence, if necessary, we may assume that $\{T(x_i)\}$ is bounded away from X, and that $\{f(x_i)\}$ converges to some point y_0 of the compact manifold M^n. Since $\{x_i\}$ converges to infinity, $f^{-1}(y_0)$ must consist of exactly one point, and $\{T(x_i)\}$ converges to $f^{-1}(y_0)$. But $f^{-1}(y_0) \in X$. This contradicts the fact that $\{T(x_i)\}$ is bounded away from X. A similar argument shows that sequences converging to X are mapped by T onto sequences converging to infinity.

(4). If V is an open neighborhood of infinity in $R^n - X$, then $X + T(V)$ is an open neighborhood of X. If W is an open neighborhood of X, then $T(W - X)$ is an open neighborhood of infinity. To prove the first assertion, note that T(V) is open. Hence we need only show that $X \subset \text{Int}(X + T(V))$. Suppose that $\{x_i\}$ is a sequence of points of $R^n - X$ converging to X. Then $\{T(x_i)\}$ eventually lies in V, by (3). Therefore $\{x_i\}$ eventually lies in T(V). Thus $X + T(V)$ is a neighborhood of X. The second assertion is proved similarly.

(5). X is cellular. To show this, let W be any open neighborhood of X. We shall exhibit an n-cell Q such that $X \subset \text{Int}\, Q \subset Q \subset W$. We may assume that W is bounded.

If S is any (n - 1)-sphere in R^n, we let B[S] and U[S] represent the open, bounded and unbounded complementary domains of S in R^n, respectively.

By (4), $T(W - X)$ is a neighborhood of infinity. Let S be a large round (n - 1)-sphere in $T(W - X)$ such that $U[S] \subset T(W - X)$. Now,

$$X + T(U[S]) \subset X + T(U[S]) + T(S) \subset W.$$

But $X + T(U[S])$ is a bounded open set containing X whose boundary consists precisely of the sphere T(S). Hence, $X + T(U[S]) = B[T(S)]$. Since S is bicollared, T(S) is bicollared. Therefore, by [1], the closed complementary domains of T(S) are closed cells. Hence $X + T(U[S]) + T(S) = \text{Cl}(B[T(S)])$ is an n-cell with the desired properties.

(6). We now construct a local homeomorphism g of S^n onto M^n that is exactly two-to-one. Such a map is known to be a covering map.

It is easy to prove (in any case, it follows from [4]) that there exists a neighborhood W of X on which f is one-to-one. Using (5), choose an n-cell Q that contains X in its interior, and such that $f \mid Q$ is one-to-one. Note that f(X) is cellular in Int f(Q). Hence, by Theorem 1 of [1], there exists a mapping ψ of M^n onto itself such that $\psi \mid M^n - f(Q) = 1$, $\psi \mid M^n - f(X)$ is a homeomorphism, and $\psi(X)$ is a single point of M^n, say y_0. Let $x_0 = (f \mid Q)^{-1}(y_0)$.

Let $g'\colon R^n \to M^n$, where

$$g' \mid Q = f \mid Q \quad \text{and} \quad g' \mid R^n - Q = \psi f \mid R^n - Q.$$

Then g' is a local homeomorphism that is exactly two-to-one on $R^n - x_0$. Extend g' to $g\colon S^n \to M^n$ by defining $g(\infty) = y_0$. Then g is one-to-one in a neighborhood of infinity, and by (3), g is continuous. Thus g is a local homeomorphism. Q.E.D.

3. AN UPPER BOUND FOR $w(M^n)$

In this section we use a sharpened form of the technique of Lemma 1 to get an upper bound on $w(M^n)$. This allows us to calculate $w(M^n)$ in a few cases.

THEOREM 3. *If* M^n *is a compact manifold, other than the* n-*sphere, that can be covered with* r *open cells, then* $w(M^n) \leq r$.

Proof. By compactness of M^n, there exist flat (closed) n-cells $Q_1, Q_2, \cdots, Q_r$ such that $M^n = Q_1 + \cdots + Q_r$. It will suffice to prove that the following proposition (P_i) holds, whenever $1 \leq i \leq r$:

(P_i) *There exists a locally one-to-one map* $f_i\colon B^n \to M^n$ *such that*

(a) $f_i(\mathring{B}^n) \supset Q_1 + \cdots + Q_i$,

(b) for each $x \in M^n$, $f_i^{-1}(x)$ consists of at most i points,

(c) There is a point $x_i \in \partial B^n$ such that $f_i(x_i) \notin Q_{i+1}$ $(i \leq r - 1)$.

We shall use the fact that, since $M^n \neq S^n$, M^n cannot be expressed as the union of two flat cells. For a discussion of this, see [5].

If N is a subset of ∂B^n and A is a set of positive reals, we shall write $N \times A = \{tx \mid t \in A,\ x \in N\}$.

We leave (P_1) to the reader, with the suggestion that he use the fact that M^n is not a sphere.

Now suppose (P_i) is known to be true $(i < r)$, and that f_i and x_i satisfy (a) to (c) above. We claim that f_i and x_i could have been chosen so that they also satisfy the following: There exists a round closed neighborhood N of x_i in ∂B^n, together with an extension of f_i to a map $g_i\colon B^n + (N \times [1, 4]) \to M^n$ such that

(i) g_i is one-to-one on some neighborhood of $N \times [1, 4]$,

(ii) $g_i(N \times [1, 4]) \subset M^n - Q_{i+1}$,

(iii) $g_i(N \times [a, b])$ is a flat cell $(1 \leq a < b \leq 4)$.

Indeed, if f_i and x_i do not satisfy these extra conditions, there exists a small number $\eta > 0$ such that $f_i \mid B^n(1 - \eta)$ and $(1 - \eta)x_i$ do satisfy (a) to (c) and (i) to (iii).

We let $B^* = B^n + (N \times [1, 4])$.

Now let h_1 be a homeomorphism of M^n onto itself that reduces to the identity on $g_i(N \times [1, 2])$, and such that

$$h_1 g_i(\text{Int } N \times (3, 4)) \supset Q_{i+1}.$$

This is possible, since $g_i(N \times [1, 2])$ and $g_i(N \times [3, 4])$ are disjoint flat cells in the complement of the flat cell Q_{i+1}.

At this stage we must make sure that the mapping under construction will behave properly with respect to Q_{i+2}. If $i + 1 = r$, let $h_2 = 1$. If $i + 1 \neq r$, we notice that, since M^n is not a sphere, there exists a point y such that

$$y \in M^n - [h_1 g_i(N \times [1, 4]) + Q_{i+2}] \subset M^n - [g_i(N \times [1, 2]) + Q_{i+1} + Q_{i+2}].$$

Let h_2 be a homeomorphism of M^n reducing to the identity on $g_i(N \times [1, 2]) + Q_{i+1}$ and taking $h_1 g_i(4x_i)$ onto y.

Define $f'_{i+1}\colon B^* \to M^n$ as follows:

$$f'_{i+1} \mid B^n = g_i \mid B^n = f_i,$$

$$f'_{i+1} \mid N \times [1, 4] = h_2 h_1 g_i \mid N \times [1, 4]\,.$$

Now f'_{i+1} is a locally one-to-one map of B^*, and it is one-to-one on $N \times [1, 4]$. Since f_i is at most i-to-one, f'_{i+1} is at most $(i + 1)$-to-one. Clearly,

$$f'_{i+1}(\text{Int } B^*) \supset Q_1 + \cdots + Q_{i+1} \quad \text{and} \quad f'_{i+1}(4x_i) \notin Q_{i+2}.$$

Let ϕ be a homeomorphism of B^n onto B^*. Let $f_{i+1} = f'_{i+1}\phi$, and let $x_{i+1} = \phi^{-1}(4x_i)$. This proves (P_{i+1}). Q. E. D.

COROLLARY 1. $w(S^n) = 3$, *whenever* $n \geq 2$.

To see this, let Q_1 and Q_3 be the upper and lower hemispheres of S^n, respectively, and let Q_2 be a flat cell in the interior of Q_1. Then the construction in the above proof gives a local homeomorphism of R^n onto S^n that is at most three-to-one. Thus $w(S^n) \leq 3$. By Theorem 2, $w(S^n) = 3$.

COROLLARY 2. *If* M^n *is a compact triangulable* n-*manifold, then* $w(M^n) \leq n + 1$.

Brown and Rosen have proved [3] that such a manifold can be covered with $n + 1$ open cells.

COROLLARY 3. *If* M^2 *is a compact* 2-*manifold, then* $w(M^2) = 3$, *unless* M^2 *is the projective plane* P^2. *Moreover,* $w(P^2) = 2$.

This follows from the triangulability of 2-manifolds, Corollary 2, and Theorem 2.

COROLLARY 4. *If* M^n *is a closed, connected, combinatorial* n-*manifold that is geometrically* $[n/r]$ - *connected* $(r \geq 3)$, *then* $w(M^n) \leq r$.

This is a consequence of Theorem 3 and of Corollary 2 of [6].

COROLLARY 5. $w(S^n \times S^m) = 3$, *whenever* $n, m \geq 1$.

$S^n \times S^m$ can be covered with three open cells. Apply Theorems 2 and 3.

4. MANIFOLDS WITH BOUNDARY

By an n-manifold with boundary we mean a separable connected metric space M^n in which each point has an open neighborhood homeomorphic to an open subset of $R^{n-1} \times [0, \infty)$ and in which $\partial M^n \neq \emptyset$. The boundary of M^n (∂M^n) is the set of all points of M^n having no neighborhood homeomorphic to R^n. The interior of M^n (Int M^n) is equal to $M^n - \partial M^n$. If ∂M^n is connected, then ∂M^n is an (n - 1)-manifold. (Recall that our definition of manifold includes connectedness.)

THEOREM 4. *If* M^n *is an* n-*manifold with nonempty connected boundary* M^{n-1}, *then there exists a local homeomorphism of* $R^{n-1} \times [0, \infty)$ *onto* M^n.

Proof. We apply Theorem 1 to get a local homeomorphism ϕ of R^{n-1} onto M^{n-1}. We may assume that $\phi \mid B^{n-1}$ is a homeomorphism. By [2] there exists a homeomorphism F of $M^{n-1} \times [0, 2)$ onto a neighborhood of M^{n-1} in M^n such that $F(m, 0) = m$ for all $m \in M^{n-1}$. Define f': $R^{n-1} \times [0, 1] \to M^n$ by setting $f'(x, t) = F(\phi(x), t)$. Then f' is a locally one-to-one mapping, and $f' \mid B^{n-1} \times [0, 1]$ is a homeomorphism.

By a slight change in the proof of Lemma 1, we can conclude that there exists a mapping f'' of $B^{n-1} \times [0, \infty)$ into M^n such that

$$f'' \mid B^{n-1} \times [0, 1] = f' \mid B^{n-1} \times [0, 1],$$

$$f''(\mathring{B}^{n-1} \times (0, \infty)) = \text{Int } M^n = f''(B^{n-1} \times (0, \infty),$$

and $f'' \mid B^{n-1} \times [2i, 2i + 3]$ is a homeomorphism $(i = 0, 1, 2, \cdots)$.

Define f: $(R^{n-1} \times [0, 1]) + (B^{n-1} \times [0, \infty)) \to M^n$ by demanding that f agree with f' and f'' on their respective domains. Let g be a homeomorphism of $R^{n-1} \times [0, \infty)$ onto $(R^{n-1} \times [0, 1)) + \mathring{B}^{n-1} \times [0, \infty))$. Then $\psi = fg$ is a local homeomorphism of $R^{n-1} \times [0, \infty)$ onto M^n. Q. E. D.

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The University of Michigan