POSITIVE EIGENVECTORS OF POSITIVE POLYNOMIAL OPERATORS ON BANACH SPACES

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1. In this note we use some results by M. A. Krasnoselskiĭ [1] to obtain two theorems on the existence of positive eigenvectors of positive polynomial operators defined in a real Banach space E with a cone K. Applications to a differential equation and an integral equation are considered at the end of the paper.

A continuous operator P from a real Banach space E into itself is called a *polynomial of degree* n if

$$\Delta_h^{n+1} P(x) = 0 \quad \text{and} \quad \Delta_h^n P(x) \neq 0$$

for all h, $x \in E$, where

$$\Delta_h^1 P(x) = P(x+h) - P(x) \quad \text{and} \quad \Delta_h^k P(x) = \Delta_h^1(\Delta_h^{k-1} P(x)) \quad (k \geq 2).$$

A polynomial of degree n can be written uniquely as

$$P(x) = U_0(x) + U_1(x) + \cdots + U_n(x),$$

where $U_k(x) = U_k^*(x, \cdots, x)$ $(k = 1, \cdots, n)$ and $U_k^*(x_1, \cdots, x_k)$ is an operator from $E \times E \times \cdots \times E$ into E that is additive and continuous in each variable x_i $(i = 1, 2, \cdots, k)$, and where $U_0(x)$ is a constant vector in E [2]. The modulus $\|P\|$ of a polynomial is defined by $\|P\| = \sup\{\|P(x)\| \mid \|x\| \leq 1\}$.

A closed set K in E is called a *cone* if

(i) $x \in K$ implies $\lambda x \in K$ for each $\lambda \geq 0$,

(ii) $x \in K$ and $y \in K$ implies $x + y \in K$,

(iii) $x \in K$ and $x \neq 0$ implies $-x \notin K$.

An element $x \in E$ is called *positive* if $x \in K$, and we write $x \geq 0$. The statement $x \geq y$ means that $x - y \in K$.

An operator T from E into E is called *positive* $(T \geq 0)$ if $T(x) \geq 0$ for each $x \geq 0$. A vector $u \in E$ $(u \neq 0)$ is an *eigenvector* of an operator T if there exists a number λ such that $T(u) = \lambda u$. The number λ is called the *eigenvalue* of T corresponding to the eigenvector u. We say that the set B of eigenvectors of T forms a *continuous branch of length* r if for each positive number r_1 $(r_1 < r)$ the intersection of B with the boundary Γ of each open set containing the zero-vector and contained in the sphere $\{x \mid \|x\| \leq r_1\}$ is nonempty.

Denoting by K_r the set $\{x \mid x \geq 0,\ \|x\| \leq r\}$, we say that an operator A from E to E is a *monotonic minorant* of an operator T on K_r if

(a) $x \leq y$ $(x, y \in E)$ implies $A(x) \leq A(y)$,

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(b) $A(x) \leq T(x)$ for each $x \in K_r$,

(c) there exist a vector $u \geq 0$ and two numbers $c > 0$, $\gamma > 0$ such that $x - \gamma u \notin K$ for each $x \in K_r$ and $cA(tu) \geq tu$ $(0 \leq t < \gamma)$.

THEOREM 1. *Let* K *be a cone in a real Banach space* E. *Let*

$$P(x) = U_1(x) + U_2(x) + \cdots + U_n(x)$$

be a completely continuous polynomial of degree n *from* E *to* E *with* $U_i \geq 0$ $(i = 1, 2, \cdots, n)$. *Assume that the linear term* U_1 *has a positive eigenvector* u *with a corresponding positive eigenvalue* μ *such that* $U_1(u) = \mu u$. *Then the set of eigenvectors of* P *forms a continuous branch of infinite length in* K.

Proof. Note first that U_1, being linear, satisfies $U_1(x) \leq U_1(y)$ whenever $x \leq y$. From the positivity of each term of P it follows that $P(x) \geq U_1(x)$ for each $x \in K$.

Let r be a positive number, and let γ be a positive number such that $x - \gamma u \notin K$ for each $x \in K_r$. Such a number γ exists, for otherwise, there would exist, for each γ (no matter how large), a vector $x \in K_r$ such that $x - \gamma u \in K$. But this would imply that $x - \gamma u = u + x - (1 + \gamma)u \geq 0$, so that

$$\frac{1}{1+\gamma} u \geq u - \frac{1}{1+\gamma} x;$$

since $\|x\| \leq r$, we would (by letting γ increase without bounds) get the contradiction that $0 \geq u$. Since

$$\frac{1}{\mu} U_1(tu) = tu \qquad \text{for } 0 \leq t < \gamma,$$

it follows that U_1 is a monotonic minorant of P on K_r. Applying a theorem by Krasnoselskiĭ [1, p. 268, Theorem 2.4], we conclude that the set B of eigenvectors of P forms a continuous branch of length r in K. Since r is arbitrary, the assertion of Theorem 1 follows.

We shall say that a positive linear operator A from E to E is u_0-bounded if there exists an element $u_0 \in K$ $(u_0 \neq 0)$ such that for each nonzero $x \in K$ there exist a positive integer n and positive numbers a, b such that $au_0 \leq A^n(x) \leq bu_0$. Krasnoselskiĭ proved that a completely continuous u_0-bounded operator has a unique unit eigenvector $u \in K$ [1, p. 261, Theorem 2.2].

THEOREM 2. *Let* K *be a cone in a real Banach space* E, *and let*

$$P(x) = U_1(x) + U_2(x) + \cdots + U_n(x)$$

be a completely continuous polynomial of degree n *from* E *to* E *with* $U_i \geq 0$ $(i = 1, 2, \cdots, n)$. *Assume that the linear term* U_1 *is* u_0*-bounded and that there exists a positive number* δ *such that* $\|U_1(x)\| \geq \delta$ *and* $\|U_n(x)\| \geq \delta$ *for each unit vector* $x \in K$. *Let* $u \in K$ *be the unique unit eigenvector of* U_1, *and* μ *the corresponding eigenvalue. Then the set of eigenvalues of* P *consists of the interval* (μ, ∞) *and possibly* μ.

Proof. Let B denote the infinite branch of eigenvectors of P in K whose existence is asserted in Theorem 1. Since $P(x) = \lambda x$ for $x \in B$, with $\lambda = \lambda(x) > 0$ depending continuously on $x \in B$, we can write, for $x \neq 0$,

$$(1)\quad \lambda(x) = \frac{1}{\|x\|}\,\|P(x)\| = \left\| U_1\left(\frac{x}{\|x\|}\right) + \|x\|\,U_2\left(\frac{x}{\|x\|}\right) + \cdots + \|x\|^{n-1} U_n\left(\frac{x}{\|x\|}\right)\right\|.$$

We show first that

$$(2)\qquad \lim_{\|x\|\to 0,\, x\in B} \lambda(x) = \mu ,$$

Let $\{x_n\}$ be any sequence of vectors in B such that $\lim_{n\to\infty} x_n = 0$. Since complete continuity of P implies complete continuity of U_1, it follows that $\{x_n\}$ contains a subsequence $\{x_k\}$ such that

$$\lim_{k\to\infty} U_1\left(\frac{x_k}{\|x_k\|}\right) = y$$

for some $y \in K$ which, by hypothesis, is not the zero-vector. Since by (1) $\lim_{k\to\infty} \lambda(x_k) = \|y\|$, it follows from

$$\left\|\frac{x_k}{\|x_k\|} - \frac{y}{\|y\|}\right\| = \left\|\frac{1}{\lambda(x_k)}\frac{P(x_k)}{\|x_k\|} - \frac{y}{\|y\|}\right\|$$

$$= \left\|\frac{1}{\lambda(x_k)} U_1\left(\frac{x_k}{\|x_k\|}\right) - \frac{1}{\lambda(x_k)} y + \frac{1}{\lambda(x_k)}\frac{P(x_k) - U_1(x_k)}{\|x_k\|} + \frac{1}{\lambda(x_k)} y - \frac{1}{\|y\|} y\right\|$$

$$\leq \frac{1}{\lambda(x_k)}\left\| U_1\left(\frac{x_k}{\|x_k\|}\right) - y\right\|$$

$$+ \frac{1}{\lambda(x_k)}\left\| \|x_k\|\, U_2\left(\frac{x_k}{\|x_k\|}\right) + \cdots + \|x_k\|^{n-1} U_n\left(\frac{x_k}{\|x_k\|}\right)\right\| + \left|\frac{1}{\lambda(x_k)} - \frac{1}{\|y\|}\right| \|y\|$$

that $\lim_{k\to\infty} \frac{x_k}{\|x_k\|} = \frac{y}{\|y\|}$.

From the continuity of U_1 it now follows that $U_1\left(\frac{y}{\|y\|}\right) = y$, that is, $U_1(y) = \|y\|\,y$. Since U_1 has only one unit eigenvector in K, we conclude that $\mu = \|y\|$, so that $\lim_{k\to\infty} \lambda(x_k) = \mu$, and (2) follows.

In view of (1), we can write

$$\lambda(x) = \|x\|^{n-1}\left\|\frac{1}{\|x\|^{n-1}} U_1\left(\frac{x}{\|x\|}\right) + \cdots + U_n\left(\frac{x}{\|x\|}\right)\right\|$$

$$\geq \|x\|^{n-1}\left(\left\| U_n\left(\frac{x}{\|x\|}\right)\right\| - \left\|\frac{1}{\|x\|^{n-1}} U_1\left(\frac{x}{\|x\|}\right) + \cdots + \frac{1}{\|x\|} U_{n-1}\left(\frac{x}{\|x\|}\right)\right\|\right)$$

$$\geq \frac{1}{2}\,\delta\,\|x\|^{n-1},$$

for sufficiently large $\|x\|$, and consequently

(3) $$\lim_{x \to \infty, x \in B} \lambda(x) = \infty .$$

We now apply a result by Krasnoselskiĭ [1, p. 274, Theorem 3.2], which states that if (2) and (3) are satisfied, then the set of eigenvalues of P contains the interval (μ, ∞).

On the other hand, if $\alpha > 0$ is any eigenvalue of P, it follows from another result by Krasnoselskiĭ [1, p. 278, Lemma 3.4] that $\alpha \geq \mu$. The theorem is proved.

2. As an application of Theorem 1, consider the differential equation

(4) $$y'' + \frac{1}{\lambda}(y + ay^2) = 0,$$

subject to the two-point boundary condition $y(0) = y(1) = 0$. Here $a = a(t)$ is a nonnegative continuous function defined on $0 \leq t \leq 1$. The equivalent integral equation is

(5) $$\int_0^1 K(s, t)(y(t) + a(t)y^2(t))\,dt = \lambda y(s),$$

where $K(s, t) = s(1 - t)$ for $0 \leq s \leq t$ and $K(s, t) = t(1 - s)$ for $t \leq s \leq 1$. If we let E be the space of all continuous functions $y(t)$ defined on $0 \leq t \leq 1$ with $y(0) = y(1) = 0$, and with norm $\|y\| = \sup\{|y(t)| \mid 0 \leq t \leq 1\}$, and if K denotes the cone $\{x \in E \mid x(t) \geq 0\}$, then the left-hand member of (5) is a polynomial of degree 2 for which the conditions of Theorem 1 are satisfied. We write this polynomial in the form

$$P(y) = U_1(y) + U_2(y) = \int_0^1 K(s, t)y(t)\,dt + \int_0^1 K(s, t)a(t)y^2(t)\,dt .$$

In K, the linear part $U_1(y)$ has the unit eigenvector $u(t) = \sin \pi t$ with corresponding eigenvalue $\lambda = 1/\pi^2$; Theorem 1 implies that the set of eigenvectors of P forms a continuous branch B of infinite length in K.

We have shown that for each positive number α there exist a nonnegative continuous function $\phi(t)$, defined on $0 \leq t \leq 1$ with $\|\phi\| = \alpha$, and a positive number μ such that $\phi(t)$ is a solution of (4) with $\lambda = \mu$ satisfying the condition $\phi(0) = \phi(1) = 0$.

As an application of Theorem 2, consider the integral equation

(6) $$\begin{aligned} P(x) &= U_1(x) + U_2(x) + \cdots + U_n(x) \\ &= \int_0^1 K_1(s, t)x(t)\,dt + \int_0^1 K_2(s, t)x^2(t)\,dt + \cdots + \int_0^1 K_n(s, t)x^n(t)\,dt = \lambda x(s). \end{aligned}$$

We let E be the space $L^1(0, 1)$, and K the cone of nonnegative functions in E. Under the assumption that the nonnegative kernels $K_1, K_2, \cdots, K_n$ are continuous on the square $0 \leq s, t \leq 1$ and that $K_i(s, t) \geq \delta$ $(i = 1, n;\ 0 \leq s, t \leq 1)$ for some positive number δ, it follows that P is a completely continuous polynomial of degree n, that $U_i \geq 0$ $(i = 1, 2, \cdots, n)$, and that $\|U_i(x)\| \geq \delta$ $(i = 1, n)$ for each $x \in K$ with $\|x\| = 1$. If we take $u_0(t) \equiv 1$ and $x \in K$, then the inequalities

$$\delta \|x\| = \delta \int_0^1 x(t)\,dt \leq \int_0^1 K_1(s, t)x(t)\,dt \leq M \int_0^1 x(t)\,dt = M\|x\|,$$

where $M = \sup\{K_1(s, t) \mid 0 \leq s, t \leq 1\}$, guarantee that U_1 is u_0-bounded. All conditions of Theorem 2 are now satisfied, and we conclude that (6) has a solution for each $\lambda > \mu$, where μ is the positive eigenvalue corresponding to the positive eigenvector of U_1.

REFERENCES

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