MONOTONE FUNCTIONS AND CONVEX FUNCTIONS

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In their note [1], A. S. Besicovitch and R. O. Davies prove the following interesting theorem concerning monotone functions of a real variable.

THEOREM. *Let* f *be a real-valued, nonnegative, continuous, monotone function defined on* $I = [0, 1]$. *Then there exist two convex functions* g_1 *and* g_2 *on* I *such that* $0 \le g_1 \le f \le g_2$ *and*

$$2 \int_0^1 g_1 \, dx \ge \int_0^1 f \, dx \ge \frac{1}{2} \int_0^1 g_2 \, dx .$$

Furthermore, the constants 2 *and* 1/2 *are best possible.*

In the present note, we extend this theorem to functions that are monotone on I^n, the n-fold cartesian product of I. Of course, the constants involved will depend upon n. The extension is an immediate corollary of Theorem A below, which concerns the measure of a certain family of subsets of euclidean n-space. The present proof is not an extension of that in [1], and in a sense it is better, since it avoids a transfinite construction.

1. THE STATEMENT OF THE MAIN THEOREM.

By R we mean the real numbers, and by P the nonnegative real numbers. $X^n = X \times \cdots \times X$ will denote the n-fold cartesian product of a set X.

For each point $x = (x^1, x^2, \cdots, x^n) \in P^n$, let

$$B_n(x) = \{y \in R^n \mid 0 \le y^i < x^i \ (i = 1, 2, \cdots, n)\},$$

$$\mathcal{M}_n = \{M \subset R^n \mid M \text{ is a bounded open set such that } x \in M \cap P^n \text{ implies } B_n(x) \subset M\}.$$

We shall partially order R^n by agreeing that $x \ge 0$ if and only if $x^i \ge 0$ $(i = 1, 2, \cdots, n)$. For each point $x \in R^n$ such that $x^i > 0$ $(i = 1, 2, \cdots, n)$, we let

$$K_n(x) = \{y \in R^n \mid \sum_{i=1}^{n} (y^i/x^i) \ge n\}.$$

Corresponding to each $M \in \mathcal{M}_n$, we define three sets $O_n(M)$, $C_n(M)$, and $H_n(M)$ as follows:

Received June 8, 1965.
The first author was supported by National Science Foundation Research Grant GP 3834.

$O_n(M) = P^n \cap M$,

$C_n(M) = P^n \setminus$ (closed convex hull of $P^n \setminus M$),

$H_n(M) = P^n \setminus \bigcap K_n(x)$, where the intersection is taken over all x in the interior of $O_n(M)$.

By $m_n[\cdot]$ we denote Lebesgue measure on R^n.

THEOREM A. *If* $M \in \mathcal{M}_n$, *then*

$$C_n(M) \subset O_n(M) \subset H_n(M) \quad \textit{and} \quad n!\, m_n[C_n(M)] \geq m_n[O_n(M)] \geq \frac{n!}{n^n} m_n[H_n(M)].$$

Furthermore, the constants $n!$ *and* $n!/n^n$ *are best possible.*

COROLLARY. *Let* $f\colon I^n \to R$ *be such that* $f \geq 0$ *and* $f(x+h) - f(x) \geq 0$ *for* $h \geq 0$ *and* $x + h \in I^n$. *Then there exist two convex functions* g_1 *and* g_2 *such that*

$$0 \leq g_1 \leq f \leq g_2 \quad \textit{and} \quad (n+1)! \int g_1 dx \geq \int f dx \geq \frac{n!}{(n+1)^n} \int g_2 dx.$$

Furthermore, the constants are best possible.

The corollary follows easily from Theorem A, which will be proved in the next section. We remark that the convex function g_2 of the corollary need not be continuous at points $x \in I^n$ with $x^i = 1$ for some $i = 1, 2, \cdots, n$. If f is continuous, then g_2 can be chosen continuous also.

2. PROOF OF THEOREM A

If $X \subset R^n$ and $t \in R$, we write

$$S(X, t, n) = \{(x^1, x^2, \cdots, x^{n-1}) \mid (x^1, x^2, \cdots, x^{n-1}, t) \in X\}.$$

LEMMA 1. *For each* $M \in \mathcal{M}_n$ *we have the relation* $S(M, t, n) \in \mathcal{M}_{n-1}$. *Furthermore,* $S(O_n(M), t, n) = O_{n-1}(S(M, t, n))$.

For each measurable set $A \subset R^{n-1}$ and for $a < b$, we let $D_n(A, a, b) = A \times [a, b)$. Clearly, $m_n[D_n(A, a, b)] = m_n[D_n(A, 0, b-a)] = (b-a)\, m_{n-1}[A]$. Next, suppose A is a measurable subset of P^{n-1} and $r > 0$. Then we let

$$E_n(A, r) = \{(x, y) \in P^{n-1} \times P \mid \exists\, \bar{x} \in A \text{ and } t \in R \text{ such that } (x, y) = t(\bar{x}, 0) + (1-t)(0, r)\},$$

and

$$F_n(A, r) = \{(x, y) \in P^{n-1} \times P \mid \exists\, \bar{x} \in A \text{ and } t \in R \text{ such that } (x, y) = t(\bar{x}, r) + (1-t)(0, nr)\}.$$

A simple calculation shows that

$$m_n[E_n(A, r)] = \frac{1}{n} m_n[D_n(A, 0, r)] = \frac{r}{n} m_{n-1}[A]$$

and

$$m_n[F_n(A, r)] = \left(\frac{n}{n-1}\right)^{n-1} m_n[D_n(A, 0, r)] = \left(\frac{n}{n-1}\right)^{n-1} r\, m_{n-1}[A].$$

Finally, let $A_1, A_2, \cdots, A_k$ be subsets of P^n. Then $\bigoplus_{i=1}^{k} A_i \equiv A_1 \oplus A_2 \oplus \cdots \oplus A_k$ is defined to be the set of points $(x^1, x^2, \cdots, x^n)$ in P^n with the following property: For some nonempty subset J of $\{1, 2, \cdots, k\}$, there exist points

$$(x^1, x^2, \cdots, x^{n-1}, x_i^n) \in A_i,$$

with $i \in J$, such that $0 \leq x^n < \sum_{i \in J} x_i^n$.

The proof of the following lemma is easy, and we omit it.

LEMMA 2. (1) *Suppose* $M_i \in \mathcal{M}_n$ $(i = 1, 2, \cdots, k)$. *Then*

$$m_n\left[\bigoplus_{i=1}^{k} O_n(M_i)\right] = \sum_{i=1}^{k} m_n[O_n(M_i)].$$

(2) *Suppose* $M \in \mathcal{M}_{n-1}$. *Then*

$$E_n(O_{n-1}(M), r) \subset D_n(O_{n-1}(M), 0, r) \subset F_n(O_{n-1}(M), r).$$

We are now ready to prove Theorem A. The inclusion relations are easily verified, since the elements M of $\mathcal{M}_n$ are open sets. The remainder of the proof is by induction. The theorem is obvious when $n = 1$. Let us assume the theorem is true for $n - 1$, and let $M \in \mathcal{M}_n$. We assume $O_n(M) \neq \square$, for otherwise the conclusion is obvious. Let

$$L = \sup\{t \in R \mid (x^1, x^2, \cdots, x^{n-1}, t) \in O_n(M)\}.$$

Then $0 < L < \infty$. Let N be a fixed positive integer, and for each integer i, let $A_i = S\left(O_n(M), \frac{i}{N} L, n\right)$. Clearly,

$$\bigoplus_{i \geq 1} D_n\left(A_i, 0, \frac{L}{N}\right) = \bigcup_{i \geq 1} D_n\left(A_i, \frac{i-1}{N} L, \frac{i}{N} L\right) \subset O_n(M)$$

$$\subset \bigcup_{i \geq 0} D_n\left(A_i, \frac{i}{N} L, \frac{i+1}{N} L\right) = \bigoplus_{i \geq 0} D_n\left(A_i, 0, \frac{L}{N}\right).$$

Furthermore,

$$\lim_{N \to \infty} \sum_{i \geq 1} m_n\left[D_n\left(A_i, \frac{i-1}{N} L, \frac{i}{N} L\right)\right] = m_n[O_n(M)]$$

$$= \lim_{N \to \infty} \sum_{i \geq 0} m_n\left[D_n\left(A_i, \frac{i}{N} L, \frac{i+1}{N} L\right)\right].$$

The sets $E_n\left(C_{n-1}(A_i), \frac{L}{N}\right)$ have the property

$$\bigoplus_{i\geq 1} E_n\left(C_{n-1}(A_i), \frac{L}{N}\right) \subset \bigoplus_{i\geq 1} D_n\left(C_{n-1}(A_i), 0, \frac{L}{N}\right) \subset O_n(M).$$

Also, $P^n \setminus \bigoplus_{i\geq 1} E_n\left(C_{n-1}(A_i), \frac{L}{N}\right)$ is a closed convex set containing $P^n \setminus M$. Hence,

$$C_n(M) \supset \bigoplus_{i\geq 1} E_n\left(C_{n-1}(A_i), \frac{L}{N}\right).$$

Since

$$m_n\left[E_n\left(C_{n-1}(A_i), \frac{L}{N}\right)\right] = \frac{N}{nL} m_{n-1}[C_{n-1}(A_i)] \geq \frac{N}{Ln!} m_{n-1}[A_i]$$
$$= \frac{1}{n!} m_n\left[D_n\left(A_i, \frac{i-1}{N} L, \frac{i}{N} L\right)\right]$$

for all i and N, we conclude that $m_n[C_n(M)] \geq \frac{1}{n!} m_n[O_n(M)]$.

Next we consider the sets $F_n\left(H_n(A_i), \frac{L}{N}\right)$. Suppose $(x^1, x^2, \cdots, x^{n-1}, x^n)$ is an interior point of $O_n(M)$. Then $S(B_n(x), t, n) \subset S(O_n(M), t, n)$. Hence

$$P^{n-1} \setminus K_{n-1}(x^1, x^2, \cdots, x^{n-1}) \subset H_{n-1}(S(O_n(M), t, n))$$

whenever $x^n > t$. A simple calculation shows that

$$P^n \setminus K_n(x) \subset \bigoplus_{i\geq 0} F_n\left(H_{n-1}(A_i), \frac{L}{N}\right).$$

Since

$$m_n\left[F_n\left(H_{n-1}(A_i), \frac{L}{N}\right)\right] = \left(\frac{n}{n-1}\right)^{n-1} \frac{N}{L} m_{n-1}[H_{n-1}(A_i)]$$
$$\leq \left(\frac{n}{n-1}\right)^{n-1} \frac{N}{L} \frac{(n-1)^{n-1}}{(n-1)!} m_{n-1}[A_i] = \frac{n^n}{n!} m_n\left[D_n\left(A_i, \frac{i}{N} L, \frac{i+1}{N} L\right)\right],$$

we see that

$$m_n\left[\bigoplus_{i\geq 0} F_n\left(H_{n-1}(A_i), \frac{L}{N}\right)\right] \leq \frac{n^n}{n!} m_n\left[\bigoplus_{i\geq 0} D_n\left(A_i, \frac{i}{N} L, \frac{i+1}{N} L\right)\right]$$

for every N. Also,

$$H_n(M) = \bigcup (P^n \setminus K_n(x)) \subset \bigoplus_{i\geq 0} F_n\left(H_{n-1}(A_i), \frac{L}{N}\right)$$

for every N, where the union is taken over all interior points x of $O_n(M)$. Thus we have shown that $m_n[H_n(M)] \leq \frac{n^n}{n!} m_n[O_n(M)]$, and the induction is complete.

To see that the constants are best possible, we need only consider the set $M = \{x \mid x \in R^n, |x^i| < 1 \ (i = 1, 2, \cdots, n)\}$.

REFERENCE

1. A. S. Besicovitch and R. O. Davies, *Two problems on convex functions*, Math. Gaz. 49 (1965), 66-69.

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