PERTURBATIONS OF A NONLINEAR VOLTERRA EQUATION

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1. INTRODUCTION

In this paper we investigate the behavior as $t \to \infty$ of the solutions of the equation

$$(1.1) \qquad x'(t) = -\int_0^t a(t-\tau)g(x(\tau))\,d\tau + \zeta(t, x(t)) \qquad \left(' = \frac{d}{dt},\ 0 \le t < \infty\right),$$

where $a(t)$, $g(x)$, $\zeta(t, x)$ are given real functions. The special case $\zeta(t, x) \equiv 0$ was considered in Levin [3]. Here, as in [3], the kernel $a(t)$ is nonnegative and decreasing, and $g(x)$ satisfies the condition $xg(x) > 0$ $(x \neq 0)$. The perturbing term $\zeta(t, x)$ is considered under a variety of hypotheses that are motivated by several applications (see Corduneanu [1] and Levin and Nohel [5]).

On integrating (1.1), one obtains the equation

$$(1.2) \qquad x(t) = -\int_0^t A(t-\tau)g(x(\tau))\,d\tau + \int_0^t \zeta(\tau, x(\tau))\,d\tau + x(0),$$

where $A(t) = \int_0^t a(\tau)\,d\tau$. The special case of (1.2),

$$(1.3) \qquad x(t) = -\int_0^t A(t-\tau)g(x(\tau))\,d\tau + \xi(t),$$

where $A(t)$, $g(x)$, and $\xi(t)$ are given functions, has been investigated under a variety of conditions different from the present ones. For example, if $A(t)$ is nonpositive (instead of nonnegative), $g(x) = x$, and $\xi(t) \geq 0$, then (1.3) is an equation of renewal type. Several authors (see for example Friedman [2] and Levin [4]) have considered the case where $A(t)$ is nonnegative and decreasing (instead of increasing).

By means of an interesting and entirely different technique, Corduneanu in [1] investigates (1.3) under hypotheses partially overlapping the present ones. Roughly speaking, the assumptions on $a(t) = A'(t)$ in [1] concern its integrability, while here they concern its monotonicity. Thus, for example, here it is possible that $a(t) \notin L_1(0, \infty)$ (indeed, in Theorem 1, it may happen that $a(t) \to a(\infty) > 0$ as $t \to \infty$), while in order to apply the results of [1] one would also have to assume that $t\,a(t) \in L_1(0, \infty)$. Concerning the perturbation term, the present hypotheses are much less restrictive. Note, for example, the $b(t)$-term in Theorem 1 and the x-dependence of $\zeta(t, x)$ in Theorems 2 and 3. A more precise comparison is rather involved, and we defer it to Section 7.

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Theorem 1 concerns the equation

$$x'(t) = -\int_0^t a(t-\tau)g(x(\tau))\,d\tau - b(t) + f(t), \tag{1.4}$$

where, in the notation of (1.1), $\zeta(t, x) = -b(t) + f(t)$ is independent of x. The hypotheses on $a(t)$ and $g(x)$, with the exception of (1.8) and the condition $g \in C'$ of (1.14), are precisely those under which the unperturbed equation was investigated in [3]. The condition (1.8) is needed here to obtain a global rather than a local result. All that is required of $f(t)$ is that it be in $L_1(0, \infty)$ and satisfy some smoothness assumptions. The conditions on $b(t)$ are to some extent determined by the method of proof. Fortunately, however, they enable one to handle perturbations that are not integrable and that occur in some of the applications.

Throughout the paper, the letter K denotes, in the usual way, a finite *a priori* bound that may vary from line to line.

THEOREM 1. (i) *Suppose*

(1.5) $a(t) \in C[0, \infty)$, $(-1)^k a^{(k)}(t) \geq 0 \quad (0 < t < \infty;\ k = 0, 1, 2)$,

(1.6) $g(x) \in C(-\infty, \infty)$, $xg(x) \geq 0$,

(1.7) $G(x) = \int_0^x g(\xi)\,d\xi \to \infty \quad (|x| \to \infty)$,

(1.8) $|g(x)| \leq K_1(1 + G(x)) \quad (|x| < \infty)$ *for some* $K_1 < \infty$,

(1.9) $b(t) \in C[0, \infty) \cap C'(0, \infty)$,

(1.10) *there exists* $c(t) \in C[0, \infty) \cap C'(0, \infty)$ *such that*

$$b^2(t) \leq a(t)c(t), \qquad (b'(t))^2 \leq a'(t)c'(t) \qquad (0 < t < \infty),$$

(1.11) $f(t) \in C[0, \infty)$, $\int_0^\infty |f(t)|\,dt < \infty$.

If $x(t)$ *is a solution of* (1.4) *on* $0 \leq t < \infty$, *then*

(1.12) $|x(t)| \leq K \quad (0 \leq t < \infty)$.

(ii) *In addition, suppose*

(1.13) $-a'''(t) \geq 0 \quad (0 < t < \infty)$,

(1.14) $xg(x) > 0 \quad (x \neq 0)$, $\quad g(x) \in C'(-\infty, \infty)$,

(1.15) $b''(t)$, $c''(t)$ *exist on* $0 < t < \infty$,

(1.16) *either* $(b''(t))^2 \leq a''(t)c''(t) \quad (0 < t < \infty)$,

or $|b'(t)|$, $|tb''(t)|$, $|c''(t)| \leq K < \infty \quad (\nu \leq t < \infty)$ *for some* K, $\nu > 0$, (1.17) $f(t) \in C'(0, \infty), \quad |f'(t)| \leq K \quad (\nu \leq t < \infty).$

Then $|x''(t)| \leq K$ $(\nu \leq t < \infty)$, *and if also* $a(t) \not\equiv a(0)$, *then*

(1.18) $$\lim_{t \to \infty} x^{(j)}(t) = 0 \quad (j = 0, 1).$$

In (1.5) and (1.13), it is assumed that $a''(t)$ and $a'''(t)$ exist and are finite at every point of $0 < t < \infty$.

In Theorem 1, the existence of a solution of (1.4) is part of the hypothesis. However, it may be seen from the proof of Theorem 1 that the K in (1.12) is an *a priori* constant. This is the key fact needed to establish existence of solutions of (1.4) on $0 \leq t < \infty$. Specifically, the bound (1.12), combined with rather obvious modifications of Lemmas 1.1 and 1.2 of Nohel [6], yields the following existence result concerning solutions of (1.4) (we state it without proof; similar existence theorems hold for the other cases of (1.1) considered below).

COROLLARY 1. *Let the hypothesis of Theorem* 1(i) *be satisfied. Then for each* x_0 *there exists a solution* $x(t)$ *of* (1.4) *on* $0 \leq t < \infty$ *such that* $x(0) = x_0$. *If in addition* $g(x)$ *satisfies a Lipschitz condition, the solution* $x(t)$ *is unique.*

From the proof of Theorem 1, one readily obtains the following result.

COROLLARY 2. (i) *If* $f(t) \equiv 0$ *in* (1.4), *then the hypotheses* (1.8), (1.11), (1.17), *and* $g \in C'$ *of* (1.14) *may be omitted in Theorem* 1, *and* $\lim_{t \to \infty} x''(t) = 0$ *added to the conclusions.*

(ii) *If* $b(t) \equiv 0$ *in* (1.4), *then* (1.9) *and* (*with* $c(t) \equiv 0$) (1.10), (1.15), (1.16) *are trivially satisfied.*

(iii) *If* $f(t) = b(t) \equiv 0$, *then Theorem* 1 (*of course, with hypotheses* (1.8), (1.9), (1.10), (1.11), (1.15), (1.16), (1.17), *and* $g \in C'$ *of* (1.14) *omitted*) *reduces to Theorem* 1 *of* [3].

(iv) *If* $f(t) \equiv 0$, $b(t) \equiv b =$ constant, *and* $a(\infty) > 0$ *in* (1.4), *then by taking* $c(t) \equiv b^2/a(\infty) \geq 0$, *one trivially satisfies conditions* (1.9), (1.10), (1.15), (1.16). (*This generalizes Theorem* 2(ii) *of* [3].)

Another illustration of Theorem 1 in which $b(t)$ need not be in $L_1(0, \infty)$ is the following. In (1.4), let

$$a(t) = (1+t)^{-\alpha}, \qquad b(t) = \lambda_1 (1+t)^{-\frac{\alpha}{2} - \varepsilon},$$

where α, ε, and λ_1 are positive constants. It is easy to show that if one chooses $c(t) = \lambda_2(1+t)^{-\varepsilon}$, where

$$\lambda_2 > \lambda_1^2 \max\left(1, \frac{\left(\frac{\alpha}{2}+\varepsilon\right)^2}{\alpha\varepsilon}, \frac{\left(\frac{\alpha}{2}+\varepsilon\right)^2\left(\frac{\alpha}{2}+\varepsilon+1\right)^2}{\alpha\varepsilon(\alpha+1)(\varepsilon+1)}\right),$$

then the hypotheses of Theorem 1 concerning $a(t)$, $b(t)$, $c(t)$ are satisfied. One may note, for example, that if $\alpha = \varepsilon = 1/2$, then $a(t) \notin L_1(0, \infty)$ and $b(t) \notin L_1(0, \infty)$ while, in contrast to Corollary 2 (iv), $a(\infty) = 0$.

If $a(t) \in L_1(0, \infty)$, a modified version of the proof of Theorem 1 establishes the following result for

$$x'(t) = -p(t, x(t)) - \int_0^t a(t-\tau)g(x(\tau))\,d\tau - b(t) + f(t). \tag{1.19}$$

COROLLARY 3. (i) *Let the hypothesis of Theorem* 1(i) *be satisfied, and let* $xp(t, x) \geq 0$, *where* $p(t, x) \in C$ $(0 \leq t < \infty,\ |x| < \infty)$. *If* $x(t)$ *is a solution of* (1.19) *on* $0 \leq t < \infty$, *then* $|x(t)| \leq K$ $(0 \leq t < \infty)$.

(ii) *In addition, let the hypothesis of Theorem* 1(ii) *be satisfied; let* $p(t, x) \in C'$; *for each* $x_1 > 0$, *let there exist* $K(x_1)$ *such that*

$$|p(t, x)|,\ |p_t(t, x)|,\ |p_x(t, x)| \leq K(x_1) \qquad (0 \leq t < \infty,\ |x| \leq x_1);$$

and let $a(t) \in L_1(0, \infty)$. *Then* (1.18) *is satisfied.*

The next theorem concerns the equation

$$x'(t) = -p(t, x(t)) - \int_0^t a(t-\tau)g(x(\tau))\,d\tau + f(t, x(t)). \tag{1.20}$$

Here $p(t, x)$, $a(t)$, and $g(x)$ satisfy essentially the same conditions as before. The new element is that the perturbation term $f(t, x)$ now depends on x. This causes us to give a result that is local with respect to initial conditions, rather than global, as the one above. Roughly speaking, the hypothesis on $f(t, x)$ is that it vanish sufficiently rapidly with respect to x, while being integrable with respect to t. After stating the result, we shall give an example showing that the hypothesis is not very restrictive and is, in fact, quite natural in view of analogous results for ordinary differential equations.

Define

$$\begin{cases} \hat{g}(x) = \max\limits_{0 \leq \xi \leq x} g(\xi) \quad (0 \leq x), \qquad \hat{g}(x) = \min\limits_{x \leq \xi \leq 0} g(\xi) \quad (x \leq 0), \\ M(x) = \max(\hat{g}(x), -\hat{g}(-x)) \qquad (x \geq 0), \\ m_1(x) = \min(G(x), G(-x)), \qquad m_2(x) = \max(G(x), G(-x)), \end{cases} \tag{1.21}$$

where $G(x) = \int_0^x g(\xi)\,d\xi$. (Note that if $g(x)$ is odd and nondecreasing, then (1.21) reduces simply to $\hat{g}(x) = g(x)$, $M(x) = g(x)$, $m_1(x) = m_2(x) = G(x)$.) In the following, $D\sigma(t)$ denotes the right-hand derivative of $\sigma(t)$ (whenever it exists).

THEOREM 2. (i) *Suppose* $0 < x_1 < \infty$, $p(t, x) \in C$, $xp(t, x) \geq 0$ $(0 \leq t < \infty,\ |x| \leq x_1)$, $a(t)$ *satisfies* (1.5), $g(x) \in C$, $xg(x) \geq 0$ $(|x| \leq x_1)$, *and* $g(x)$ *is not identically zero in any neighborhood of the origin. Let* $f(t, x) \in C$ $(0 \leq t < \infty,\ |x| \leq x_1)$, *and for each* $\varepsilon > 0$ *let there exist a* $\delta = \delta(\varepsilon) > 0$, *where* $\delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *and a* $\beta(t) = \beta(t, \varepsilon) \geq 0$, *where* $\int_0^\infty \beta(t)\,dt \leq \varepsilon$, *such that* $|f(t, x)| \leq \beta(t)$ *whenever* $0 \leq t < \infty$ *and* $|x| \leq \delta$. *Moreover, let*

$$\frac{m_1(\delta(\varepsilon))}{\varepsilon M(\delta(\varepsilon))} > e \tag{1.22}$$

for sufficiently small positive ε.

Then for any $0 < x_2 \leq x_1$ *there exists an* $x_0 = x_0(x_2) > 0$ *such that every solution* $x(t)$ *of* (1.20) *on* $0 \leq t < \infty$ *with* $|x(0)| \leq x_0$ *satisfies the condition* $|x(t)| < x_2$ $(0 \leq t < \infty)$.

(ii) *In addition, let* $p(t, x) \in C'$ *and*

$$|p(t, x)|,\ |p_t(t, x)|,\ |p_x(t, x)| \leq K \quad (0 \leq t < \infty,\ |x| \leq x_1).$$

Let $a(t)$ *satisfy* (1.13) *and* $a(t) \in L_1(0, \infty)$, $a(t) \not\equiv 0$. *Let* $xg(x) > 0$ $(x \neq 0)$, $g(x) \in C'$ $(|x| \leq x_1)$. *Let* $f(t, x) \in C'$ *and*

$$|f_t(t, x)|,\ |f_x(t, x)| \leq K \quad (0 \leq t < \infty,\ |x| \leq x_1).$$

Finally, let $\beta(t)$, $D\beta(t) \leq K(\varepsilon) < \infty$ $(0 \leq t < \infty,\ 0 < \varepsilon \leq 1)$. *Then* $\lim_{t \to \infty} x^{(j)}(t) = 0$ $(j = 0, 1)$.

It is easily seen from the hypothesis that $p(t, 0) = f(t, 0) \equiv 0$. Thus Theorem 2(i) asserts that the solution $x(t) \equiv 0$ of (1.20) is stable in the sense of Ljapunov, and Theorem 2(ii) asserts that it is asymptotically stable.

To illustrate the condition (1.22), consider the following example. Let $g(x) = x^{2n+1}$ for some integer $n \geq 0$. (For simplicity, we have taken $g(x)$ odd and monotonic; more complicated functions could be treated in much the same manner.) Let there exist a function $\rho(t) \geq 0$ and a real number $m > 1$ such that

$$|f(t, x)| \leq |x|^m \rho(t) \quad (0 \leq t < \infty,\ |x| \leq x_1),$$

where

$$0 < \int_0^\infty \rho(t)\, dt < \infty; \qquad \rho(t),\ D\rho(t) \leq K \quad (0 \leq t < \infty).$$

Then

$$m_1(x) = G(x) = \frac{x^{2n+2}}{2n+2}, \qquad M(x) = x^{2n+1}.$$

Define

$$\delta = \delta(\varepsilon) = \left(\varepsilon \Big/ \int_0^\infty \rho(t)\, dt\right)^{1/m}, \qquad \beta(t) = \beta(t, \varepsilon) = \varepsilon\, \rho(t) \Big/ \int_0^\infty \rho(t)\, dt.$$

Then, clearly,

$$\int_0^\infty \beta(t, \varepsilon)\, dt = \varepsilon, \qquad |f(t, x)| \leq \delta^m \rho(t) = \beta(t, \varepsilon) \quad (|x| \leq \delta).$$

Finally,

$$\frac{m_1(\delta)}{\varepsilon M(\delta)} \doteq \varepsilon^{-1+1/m}(2n+2)^{-1}\left(\int_0^\infty \rho(t)\,dt\right)^{-1/m} \to \infty \qquad (\varepsilon \to 0),$$

so that condition (1.22) is satisfied.

The perturbation term $f(t, x)$ in (1.20), rather than being small with respect to x, in the sense of Theorem 2, may be small because of the presence of a small parameter. Therefore, consider the equation

$$x'(t) = -p(t, x(t)) - \int_0^t a(t-\tau)g(x(\tau))\,d\tau + \mu f(t, x(t)), \tag{1.23}$$

where μ is a real parameter. The following theorem plays essentially the same role with respect to Theorem 2 as small-parameter-perturbation theorems play with respect to stability theorems in ordinary differential equations.

THEOREM 3. (i) *Suppose* $0 < x_1 < \infty$, $p(t, x) \in C$, $xp(t, x) \geq 0$ $(0 \leq t < \infty$, $|x| \leq x_1)$, $a(t)$ *satisfies* (1.5), *and* $g(x) \in C$, $xg(x) \geq 0$ $(|x| \leq x_1)$. *Let* $f(t, x) \in C$ $(0 \leq t < \infty$, $|x| \leq x_1)$, *and let there exist* $\beta(t) \in L_1(0, \infty)$ *such that* $|f(t, x)| \leq \beta(t)$ $(0 \leq t < \infty$, $|x| \leq x_1)$.

Then for any $0 < x_2 \leq x_1$ *there exist* $x_0 = x_0(x_2) > 0$ *and* $\mu_0 = \mu_0(x_2) > 0$ *such that every solution* $x(t) = x(t, \mu)$ *of* (1.23) *on* $0 \leq t < \infty$ *with* $|x(0)| \leq x_0$ *and* $|\mu| \leq \mu_0$ *satisfies the condition* $|x(t)| < x_2$ $(0 \leq t < \infty)$.

(ii) *In addition, let* $p(t, x) \in C'$ *and*

$$|p(t, x)|,\ |p_t(t, x)|,\ |p_x(t, x)| \leq K \qquad (0 \leq t < \infty,\ |x| \leq x_1).$$

Let $a(t)$ *satisfy* (1.13) *and* $a(t) \in L_1(0, \infty)$, $a(t) \not\equiv 0$. *Let* $xg(x) > 0$ $(x \neq 0)$, $g(x) \in C'$ $(|x| \leq x_1)$. *Let* $f(t, x) \in C'$ *and*

$$|f_t(t, x)|,\ |f_x(t, x)| \leq K \qquad (0 \leq t < \infty,\ |x| \leq x_1).$$

Finally let $\beta(t)$, $D\beta(t) \leq K$ $(0 \leq t < \infty)$. *Then* $\lim_{t \to \infty} x^{(j)}(t) = 0$ $(j = 0, 1)$.

(Note that $\beta(t)$ of Theorem 3, unlike $\beta(t, \varepsilon)$ of Theorem 2, does not depend on ε.)

2. PRELIMINARIES

In the proofs we shall need the following lemmas.

LEMMA 1. (i) *Let* $a(t)$ *satisfy* (1.5). *Then* $t a'(t) \to 0$ $(t \to 0+)$ *and* $a'(t)$, $t a''(t) \in L_1(0, \infty)$.

(ii) *If also* (1.13) *is satisfied, then* $t^2 a''(t) \to 0$ $(t \to 0+)$ *and* $t^2 a'''(t) \in L_1(0, \infty)$.

The proof of Lemma 1 may be found in [3, Lemma 3].

LEMMA 2. *Let* $q(t) \in C'$, *and let* $Dq'(t)$ *exist on* $T \leq t < \infty$ *for some* $0 < T < \infty$. *If*

$$q(t) \geq 0, \quad q'(t) \leq 0, \quad Dq'(t) \geq -K > -\infty \qquad (T \leq t < \infty),$$

then $q'(t) \to 0$ $(t \to \infty)$. (This extends Lemma 1 of [3], where it is assumed that $q''(t)$ exists and $q''(t) \geq -K > -\infty$.)

Proof of Lemma 2. If $q'(t) \not\to 0$ as $t \to \infty$, then, since $q'(t) \leq 0$, there exist a $\lambda > 0$ and a sequence $\{t_n\}$, where $t_n \to \infty$ as $n \to \infty$, such that $q'(t_n) \leq -\lambda < 0$. For $n \geq N$ (N sufficiently large), consider the intervals

$$I_n = \left[t_n - \frac{\lambda}{2K}, t_n\right], \quad \text{where } t_n - \frac{\lambda}{2K} \geq T,$$

and where K is the same as in the hypothesis. Since $Dq'(t) \geq -K$, it follows from a known property of the right-hand derivate (see [7, p. 355]) that

$$q'(t_n) - q'(t) \geq -K(t_n - t) \quad \text{when } t \in I_n,$$

and therefore

$$q'(t) \leq q'(t_n) + \frac{\lambda}{2} \leq -\frac{\lambda}{2} \quad (t \in I_n,\ n \geq N).$$

Applying the mean-value theorem and using this estimate, we see that

$$q\left(t_n - \frac{\lambda}{2K}\right) - q(t_n) \geq \frac{\lambda}{2} \cdot \frac{\lambda}{2K} = \frac{\lambda^2}{4K} \quad (n \geq N),$$

which contradicts $q(t) \downarrow q(\infty) \geq 0$ as $t \to \infty$; this completes the proof.

We observe that hypotheses (1.5) and (1.13) do not preclude the possibility that $a'(0)$, $a''(0)$, $a'''(0)$ are infinite. For this reason, some care is required in handling certain integrals that arise in the proofs of the theorems. For example, if $x(t) \in C[0, \infty)$, then

$$\frac{d}{dt} \int_0^t a'(t - \tau)\left(\int_\tau^t g(x(s))\,ds\right)^2 d\tau$$
$$= 2g(x(t)) \int_0^t a'(t - \tau)\left(\int_\tau^t g(x(s))\,ds\right) d\tau + \int_0^t a''(t - \tau)\left(\int_\tau^t g(x(s))\,ds\right)^2 d\tau.$$

In this and similar situations, Lemma 1 above and Lemma 4 of [3] justify the calculations.

3. PROOF OF THEOREM 1

(i) For $0 \leq t < \infty$, define

$$E(t) = G(x(t)) + \frac{1}{2} a(t)\left(\int_0^t g(x(s))\,ds\right)^2 + b(t)\int_0^t g(x(s))\,ds + \frac{1}{2} c(t) - \frac{1}{2}\int_0^t a'(t - \tau)\left(\int_\tau^t g(x(s))\,ds\right)^2 d\tau, \tag{3.1}$$

$$F(t) = \int_0^t |f(\tau)| \, d\tau , \tag{3.2}$$

$$V(t) = (1 + E(t)) \exp(-K_1 F(t)) . \tag{3.3}$$

From (1.5), (1.6), (1.10) it is evident that $E(t), V(t) \geq 0$. Differentiation of (3.3) yields, after some calculation involving an integration by parts (note the remark in Section 2 concerning the validity of these calculations),

$$\begin{aligned} V'(t) = -K_1 |f(t)| V(t) + \Big\{ & g(x(t)) f(t) + \frac{1}{2} a'(t) \Big(\int_0^t g(x(s))\, ds \Big)^2 + b'(t) \int_0^t g(x(s))\, ds \\ & + \frac{1}{2} c'(t) - \frac{1}{2} \int_0^t a''(t-\tau) \Big(\int_\tau^t g(x(s))\, ds \Big)^2 d\tau \Big\} \exp(-K_1 F(t)) . \end{aligned} \tag{3.4}$$

Hence from (1.5), (1.10), (3.4) we see that

$$V'(t) \leq \{ -K_1 - K_1 G(x(t)) + |g(x(t))| \} \, |f(t)| \exp(-K_1 F(t)) ,$$

which together with (1.8) implies $V'(t) \leq 0$. Therefore

$$G(x(t)) \exp\{-K_1 F(t)\} \leq V(t) \leq V(0) = 1 + G(x(0)) + \frac{1}{2} c(0) ,$$

and so

$$G(x(t)) \leq \Big\{ 1 + G(x(0)) + \frac{1}{2} c(0) \Big\} \exp\Big(K_1 \int_0^\infty |f(t)| \, dt \Big) ,$$

which together with (1.7) and (1.11) yields (1.12).

(ii) Differentiating (1.4), one obtains

$$x''(t) = -a(0) g(x(t)) - \int_0^t a'(t-\tau) g(x(\tau)) \, d\tau - b'(t) + f'(t) . \tag{3.5}$$

From (1.16) it follows that $|b'(t)| \leq K < \infty$ $(\nu \leq t < \infty)$, for in the second alternative it is assumed, and in the first alternative we can proceed as follows. Since $a''(t) \geq 0$, we see that $c''(t) \geq 0$ and therefore $-c'(t)$ is nonincreasing, which together with the last condition in (1.10) proves the assertion.

From (1.12), (1.17), (3.5), and the hypothesis (note that $a'(t) \in L_1(0, \infty)$), we see that $|x''(t)| \leq K < \infty$ $(\nu \leq t < \infty)$. This, together with (1.12) and the mean-value theorem, yields $|x'(t)| \leq K < \infty$ $(\nu \leq t < \infty)$. While we don't use it, the last inequality together with (1.4), (1.9), (1.11) implies $|x'(t)| \leq K$ $(0 \leq t < \infty)$.

Taking the right-hand derivative of $V'(t)$, one obtains after some calculation

$$DV'(t) = \Omega_1(t) + \left\{ \frac{1}{2} a''(t) \left(\int_0^t g(x(s))\,ds \right)^2 \right. \tag{3.6}$$
$$\left. + b''(t) \int_0^t g(x(s))\,ds + \frac{1}{2} c''(t) \right\} \exp(-K_1 F(t)),$$

where

$$\Omega_1(t) = -K_1 |f(t)|\, V'(t) - K_1 V(t) D|f(t)| + \left\{ g'(x(t))x'(t)f(t) + g(x(t))[-K_1 f(t)\,|f(t)| \right.$$
$$+ 2f'(t) - x''(t) - a(0)g(x(t))] - K_1 |f(t)| \left(\frac{1}{2} a'(t) \left(\int_0^t g(x(s))\,ds \right)^2 \right.$$
$$\left. + b'(t) \int_0^t g(x(s))\,ds + \frac{1}{2} c'(t) - \frac{1}{2} \int_0^t a''(t-\tau) \left(\int_\tau^t g(x(s))\,ds \right)^2 d\tau \right)$$
$$\left. - \frac{1}{2} \int_0^t a'''(t-\tau) \left(\int_\tau^t g(x(s))\,ds \right)^2 d\tau \right\} \exp(-K_1 F(t)).$$

There exists a K such that $\Omega_1(t) \geq -K > -\infty$ $(\nu \leq t < \infty)$. This follows from the conditions $V(t) \geq 0$ and $V'(t) \leq 0$, the boundedness of $x(t)$, $x'(t)$, and $x''(t)$ on $\nu \leq t < \infty$, the relation $|D|f(t)|| = |f'(t)|$, and the hypothesis. Hence, (1.16) and (3.6) imply that $DV'(t) \geq -K > -\infty$ $(\nu \leq t < \infty)$. By Lemma 2, $V'(t) \to 0$ as $t \to \infty$.

Returning to (3.4), we find as a consequence of $V'(t) \to 0$, (1.5), (1.8), (1.10), and (1.11), that

$$\lim_{t\to\infty} \int_0^t a''(t-\tau) \left(\int_\tau^t g(x(s))\,ds \right)^2 d\tau = 0. \tag{3.7}$$

The argument of [3] enables us to conclude from (3.7) that $x(t) \to 0$ $(t \to \infty)$. (This argument uses (1.5), (1.13), (1.14), the condition $a(t) \not\equiv a(0)$, and the boundedness of $x(t)$ and $x'(t)$ on $\nu \leq t < \infty$.) From the property that $x(t) \to 0$ as $t \to \infty$, the boundedness of $x''(t)$, and the mean-value theorem, we deduce that $x'(t) \to 0$ as $t \to \infty$, which completes the proof.

4. PROOF OF COROLLARY 3

Since this proof is quite similar to that in Section 3, we only indicate the differences in the two.

(i) Define $E(t)$, $F(t)$, $V(t)$ by formulas (3.1), (3.2), (3.3). Instead of (3.4), we now obtain

$$
\begin{aligned}
V'(t) = -K_1 |f(t)| V(t) + \Big\{ &- g(x(t))p(t, x(t)) + g(x(t))f(t) \\
(4.1) \qquad &+ \tfrac{1}{2} a'(t)\left(\int_0^t g(x(s))\,ds\right)^2 + b'(t)\int_0^t g(x(s))\,ds + \tfrac{1}{2} c'(t) \\
&- \tfrac{1}{2}\int_0^t a''(t-\tau)\left(\int_\tau^t g(x(s))\,ds\right)^2 d\tau \Big\} \exp(-K_1 F(t)).
\end{aligned}
$$

From (1.6) and $xp(t, x) \geq 0$, one now concludes, as in Section 3, that $V'(t) \leq 0$, which in turn yields $|x(t)| \leq K$ $(0 \leq t < \infty)$.

(ii) Since $a(t) \in L_1(0, \infty)$ and $|x(t)| \leq K$ $(0 \leq t < \infty)$, it now follows from (1.19) and the hypothesis that $|x'(t)| \leq K$ $(0 \leq t < \infty)$. The inapplicability (note the term $x'(t)$ in (4.2) below) of the indirect argument of Section 3 to show that $|x'(t)| \leq K$ $(0 \leq t < \infty)$ is the reason for the hypothesis $a(t) \in L_1(0, \infty)$.

Differentiating (1.19), one obtains

$$
\begin{aligned}
x''(t) = &-p_t(t, x(t)) - p_x(t, x(t))x'(t) - a(0)g(x(t)) \\
(4.2) \qquad &- \int_0^t a'(t-\tau)g(x(\tau))\,d\tau - b'(t) + f'(t),
\end{aligned}
$$

which, with the hypothesis and the boundedness of $x(t)$ and $x'(t)$ on $0 \leq t < \infty$, implies $|x''(t)| \leq K$ $(\nu \leq t < \infty)$.

Taking the right-hand derivative of $V'(t)$, one obtains the formula

$$
\begin{aligned}
DV'(t) = \Omega_2(t) \\
+ \left\{ \tfrac{1}{2} a''(t)\left(\int_0^t g(x(s))\,ds\right)^2 + b''(t)\int_0^t g(x(s))\,ds + \tfrac{1}{2} c''(t) \right\} \exp(-K_1 F(t)),
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega_2(t) = \Omega_1(t) + \{ &-g'(x(t))x'(t)p(t, x(t)) + g(x(t))[K_1 |f(t)| p(t, x(t)) \\
&- 2p_t(t, x(t)) - 2p_x(t, x(t))x'(t)] \} \exp(-K_1 F(t)),
\end{aligned}
$$

and where $\Omega_1(t)$ is given by the same expression as in Section 3. Again, we can show that $DV'(t) \geq -K$ $(\nu \leq t < \infty)$, and the proof is completed as in Section 3.

5. PROOF OF THEOREM 2

(i) Let $0 < x_2 \leq x_1$. Choose $\varepsilon = \varepsilon(x_2) > 0$ so that $0 < \delta(\varepsilon) \leq x_2$ and so that (1.22) is satisfied; ε, and therefore also δ and $\beta(t) = \beta(t, \varepsilon)$, are fixed for the remainder of the proof. Now choose $x_0 = x_0(x_2) > 0$ so that

$$
(5.1) \qquad (\varepsilon M(\delta) + m_2(x_0))e < m_1(\delta),
$$

which by the definitions of m_1 and m_2 implies that $x_0 < \delta$.

Let $x(t)$ be a solution of (1.20) on $0 \leq t < \infty$, with $|x(0)| \leq x_0$. Then, by continuity, $|x(t)| < \delta$ for sufficiently small t. Define

$$E(t) = G(x(t)) + \frac{1}{2} a(t)\left(\int_0^t g(x(s))\,ds\right)^2 - \frac{1}{2}\int_0^t a'(t-\tau)\left(\int_\tau^t g(x(s))\,ds\right)^2 d\tau, \tag{5.2}$$

$$V(t) = (\varepsilon M + E(t))\exp\left(-\frac{1}{\varepsilon}\int_0^t \beta(\tau)\,d\tau\right), \tag{5.3}$$

where $M = M(\delta)$.

Differentiating (5.3), we obtain

$$\begin{aligned} V'(t) = -\frac{1}{\varepsilon}\beta(t)V(t) + \Bigg\{ & g(x(t))[-p(t, x(t)) + f(t, x(t))] + \frac{1}{2} a'(t)\left(\int_0^t g(x(s))\,ds\right)^2 \\ & - \frac{1}{2}\int_0^t a''(t-\tau)\left(\int_\tau^t g(x(s))\,ds\right)^2 d\tau \Bigg\} \exp\left(-\frac{1}{\varepsilon}\int_0^t \beta(\tau)\,d\tau\right). \end{aligned} \tag{5.4}$$

Thus, by hypothesis, for as long as $|x(t)| \leq \delta$ (note, $|x(0)| \leq x_0 < \delta$), one has the relation

$$V'(t) \leq \{-M\beta(t) + |g(x(t))|\,|f(t, x(t))|\}\exp\left(-\frac{1}{\varepsilon}\int_0^t \beta(\tau)\,d\tau\right) \leq 0$$

and hence also

$$G(x(t)) \leq [\varepsilon M + G(x(0))]\exp\left(\frac{1}{\varepsilon}\int_0^t \beta(\tau)\,d\tau\right) \leq (\varepsilon M + m_2(x_0))e < m_1(\delta). \tag{5.5}$$

Suppose there exists a positive T such that $|x(T)| = \delta$. Then from (5.1), (5.5) it follows that $m_1(\delta) \leq G(x(t)) < m_1(\delta)$, which is impossible. Hence, no such T exists, and $|x(t)| < \delta \leq x_2$ $(0 \leq t < \infty)$.

(ii) Since $a(t) \in L_1(0, \infty)$, it follows from (i), (1.20), and the hypothesis that $|x'(t)| \leq K$ $(0 \leq t < \infty)$. This, together with (i), the hypothesis, and

$$\begin{aligned} x''(t) = & -p_t(t, x(t)) - p_x(t, x(t))x'(t) - a(0)g(x(t)) \\ & - \int_0^t a'(t-\tau)g(x(\tau))\,d\tau + f_t(t, x(t)) + f_x(t, x(t))x'(t), \end{aligned}$$

implies $|x''(t)| \leq K$ $(0 \leq t < \infty)$.

Taking the right-hand derivative of (5.4), one obtains the formula

$$DV'(t) = \frac{1}{2} a''(t) \left(\int_0^t g(x(s))\,ds \right)^2 \exp\left(-\frac{1}{\varepsilon} \int_0^t \beta(\tau)\,d\tau \right) - \frac{1}{\varepsilon} \beta(t) V'(t) - \frac{1}{\varepsilon} V(t) D\beta(t)$$

$$+ \Bigg\{ g'(x(t))\, x'(t) [f(t, x(t)) - p(t, x(t))]$$

$$+ g(x(t)) \left[\frac{1}{\varepsilon}\beta(t) [p(t, x(t)) - f(t, x(t))] - 2p_t(t, x(t)) - 2p_x(t, x(t))\, x'(t) \right.$$

$$\left. + 2f_t(t, x(t)) + 2f_x(t, x(t))\, x'(t) - x''(t) - a(0) g(x(t)) \right]$$

$$- \frac{1}{\varepsilon} \beta(t) \left(\frac{1}{2} a'(t) \left(\int_0^t g(x(s))\,ds \right)^2 - \frac{1}{2} \int_0^t a''(t - \tau) \left(\int_\tau^t g(x(s))\,ds \right)^2 d\tau \right)$$

$$- \frac{1}{2} \int_0^t a'''(t - \tau) \left(\int_\tau^t g(x(s))\,ds \right)^2 d\tau \Bigg\} \exp\left(-\frac{1}{\varepsilon} \int_0^t \beta(\tau)\,d\tau \right).$$

Reasoning as in the proof of Theorem 1, we deduce that $DV'(t) \geq -K > -\infty$ $(0 \leq t < \infty)$ and $V'(t) \to 0$ $(t \to \infty)$. Hence it follows from (5.4) and the hypothesis that (3.7) again holds. The proof is concluded as in Section 3.

6. PROOF OF THEOREM 3

(i) This proof is similar to the one given in Section 5. Define m_1, m_2, M by (1.21). Choose $x_0 = x_0(x_2) > 0$ so that $m_2(x_0) < m_1(x_2)$. Thus $x_0 < x_2$. Now choose $0 < \lambda \leq 1$ so that $\lambda + m_2(x_0) < m_1(x_2)$. Finally, choose $\mu_0 > 0$ so that

$$(\lambda + m_2(x_0)) \exp\left(\frac{M}{\lambda} \mu_0 \int_0^\infty \beta(\tau)\,d\tau \right) < m_1(x_2), \tag{6.1}$$

where $M = M(x_1)$.

Let $x(t) = x(t, \mu)$ be a solution of (1.23) on $0 \leq t < \infty$, with $|x(0)| \leq x_0$, $|\mu| \leq \mu_0$. By continuity, $|x(t)| < x_2$ for sufficiently small t. Define

$$V(t) = (\lambda + E(t)) \exp\left(-\frac{M\mu_0}{\lambda} \int_0^t \beta(\tau)\,d\tau \right),$$

where $E(t)$ is defined by (5.2). Then

$$V'(t) = -\frac{M\mu_0\, \beta(t)}{\lambda} V(t) + \Bigg\{ g(x(t)) [-p(t, x(t)) + \mu f(t, x(t))] + \frac{1}{2} a'(t) \left(\int_0^t g(x(s))\,ds \right)^2$$

$$- \frac{1}{2} \int_0^t a''(t - \tau) \left(\int_\tau^t g(x(s))\,ds \right)^2 d\tau \Bigg\} \exp\left(-\frac{M\mu_0}{\lambda} \int_0^t \beta(\tau)\,d\tau \right).$$

Hence, as $|\mu| \leq \mu_0$, for as long as $|x(t)| \leq x_2$ one has $V'(t) \leq 0$ and thus also

$$(6.2) \qquad G(x(t)) \leq (\lambda + m_2(x_0)) \exp\left(\frac{M\mu_0}{\lambda} \int_0^\infty \beta(\tau)\, d\tau\right) < m_1(x_2).$$

Suppose there exists $0 < T < \infty$ such that $|x(T)| = x_2$. Then, by (6.1) and (6.2), $m_1(x_2) \leq G(x(T)) < m_1(x_2)$, which is impossible. Therefore $|x(t)| < x_2$ $(0 \leq t < \infty)$.

(ii) As in the proof of Theorem 2(ii), we can now complete the proof by showing that $|x'(t)|$, $|x''(t)|$, $-DV'(t) \leq K$ $(0 \leq t < \infty)$.

7. REMARKS ON SOME RESULTS OF CORDUNEANU

In the comparison of the preceding results with those of Corduneanu [1], the case of (1.1) that must be considered is (as we shall see from the sequel)

$$(7.1) \qquad x'(t) = -\theta\, g(x(t)) - \int_0^t a(t-\tau) g(x(\tau))\, d\tau + f(t),$$

where θ is a constant and $g(x)$, $a(t)$, $f(t)$ are prescribed functions. This may be regarded either as a special case of (1.19) with $b(t) \equiv 0$ and $p(t, x) \equiv p(x) = \theta g(x)$, or of (1.20) with $f(t, x) \equiv f(t)$ and $p(t, x) \equiv p(x) = \theta g(x)$. Clearly, (7.1) is equivalent to

$$(7.2) \qquad x(t) = x(0) + F(t) - \int_0^t [A(t-\tau) + \theta] g(x(\tau))\, d\tau,$$

where

$$(7.3) \qquad A(t) = \int_0^t a(\tau)\, d\tau, \qquad F(t) = \int_0^t f(\tau)\, d\tau.$$

Corduneanu considers the equation

$$(7.4) \qquad x(t) = z(t) + \int_0^t k(t-\tau) g(x(\tau))\, d\tau,$$

which is of the same form as (7.2). He shows that *if*

$$(7.5) \qquad z'(t),\ z''(t) \in L_1(0, \infty) \cap L_2(0, \infty),$$

(7.6) $k(t) = h(t) - \rho$, *where* $\rho > 0$ *and* $h(t), h'(t) \in L_1(0, \infty) \cap L_2(0, \infty)$,

$$(7.7) \qquad g(x) \in C(-\infty, \infty), \qquad xg(x) > 0 \quad (x \neq 0),$$

$$\text{(7.8)}\quad \begin{cases} \textit{there exists a constant } q \geq 0 \textit{ such that} \\ \Omega(\omega) = \Re\{(1+i\omega q)J(i\omega)\} \leq 0 \quad (\omega \neq 0), \textit{ where} \\ H(i\omega) = \int_0^\infty h(t)\exp(-i\omega t)\,dt, \quad J(i\omega) = H(i\omega) - \frac{\rho}{i\omega}, \end{cases}$$

then every solution $x(t)$ *of* (7.4) *tends to* 0 *as* $t \to \infty$.

Thus one sees, as we noted in Section 1, that less is assumed here than in [1] about the terms $p(t, x)$ and $f(t, x)$. However, the most interesting relations between the two results are found in the comparison of the hypotheses on the respective kernels. Turning to the latter, we begin with a lemma that enables us to relate (7.2) to (7.4) and, in particular, (7.6) to $a(t)$ and θ.

LEMMA 7.1. *Let* θ *be a constant, and let* $a(t) \in C[0, \infty)$. *Then there exist a function* $h(t)$ *and a constant* ρ *such that*

$$\text{(7.9)}\qquad h(t),\ h'(t) \in L_1(0, \infty) \cap L_2(0, \infty); \qquad \rho > 0,$$

$$\text{(7.10)}\qquad h(t) - \rho = -\int_0^t a(\tau)\,d\tau - \theta,$$

if and only if

$$\text{(7.11)}\qquad a(t), \int_t^\infty a(\tau)\,d\tau \in L_1(0, \infty) \cap L_2(0, \infty); \qquad \int_0^\infty a(\tau)\,d\tau + \theta > 0,$$

$$\text{(7.12)}\qquad h(t) = \int_t^\infty a(\tau)\,d\tau, \qquad \rho = \int_0^\infty a(\tau)\,d\tau + \theta.$$

Proof. (i) If (7.11) and (7.12) are satisfied, then obviously (7.9) and (7.10) hold.

(ii) Let there exist $h(t)$ and ρ satisfying (7.9) and (7.10). From (7.10) one sees that $a(t) = -h'(t)$, which together with (7.9) implies $a(t) \in L_1 \cap L_2$. However, $h(t), h'(t) = -a(t) \in L_1$ implies $\lim_{t\to\infty} h(t) = h(\infty) = 0$. Thus, by (7.9) and (7.10),

$$\rho = \int_0^\infty a(\tau)\,d\tau + \theta > 0.$$

Substituting this formula for ρ into (7.10), one obtains $h(t) = \int_t^\infty a(\tau)\,d\tau$. This and (7.9) imply $\int_t^\infty a(\tau)\,d\tau \in L_1 \cap L_2$, which completes the proof.

If $a(t) \geq 0$, then the condition $\int_t^\infty a(\tau)\,d\tau \in L_1(0, \infty)$ of (7.11) can be stated somewhat differently:

LEMMA 7.2. *If* $a(t) \in C[0, \infty)$, $a(t) \geq 0$, $a(t) \in L_1(0, \infty)$, *and*

$$\int_t^\infty a(\tau)\,d\tau \in L_1(0, \infty),$$

then $t\,a(t) \in L_1(0, \infty)$.

The next lemma shows that in problems that can be treated by both methods, $\theta \geq 0$ (that is, $\theta g(x)$ satisfies the same sign hypothesis as a term of type $p(t, x)$).

LEMMA 7.3. *Suppose* $a(t) \in C[0, \infty)$, $a(t) \geq 0$, *and* $a(t)$ *is decreasing. Further, let there exist* $h(t)$ *and* ρ *such that* (7.8), (7.9), (7.10) *are satisfied. Then* $\theta \geq 0$.

Proof. By Lemma 7.1, $h(t) = \int_t^\infty a(\tau)\,d\tau$. Substituting this relation into the formula for $\Omega(\omega)$ in (7.8), we obtain, after an integration by parts and an application of (7.12), the formula

$$\Omega(\omega) = \frac{1}{\omega}\int_0^\infty a(t)\sin\omega t\,dt - q\int_0^\infty a(t)\cos\omega t\,dt - q\theta. \tag{7.13}$$

If $a(t) \equiv 0$, then $\rho = \theta$ by (7.12), and hence $\theta > 0$ by (7.9).

If $a(t) \not\equiv 0$, then $q > 0$. For otherwise, $q = 0$ by (7.8), and since $a(t)$ is nonnegative and decreasing, it follows from (7.13) that $\Omega(\omega) > 0$, which contradicts (7.8). Thus $q > 0$. By (7.13) and the Riemann-Lebesgue lemma,

$$\Omega(\omega) = o(1) - q(o(1) + \theta) \qquad (|\omega| \to \infty).$$

Hence, since $q > 0$, it follows from (7.8) that $\theta \geq 0$.

The next lemma shows that if $a(t)$ satisfies the monotonicity conditions (1.5) and (1.13), and also the additional integrability conditions (7.11), then conditions (7.6) and (7.8) hold.

LEMMA 7.4. *Let* $\theta \geq 0$, *and let* $a(t)$ *satisfy* (1.5), (1.13), *and* (7.11). *Define* $h(t)$ *and* ρ *by* (7.12). *Then* (7.6) *and* (7.8) *hold.* (*If* $a'(0) > -\infty$, *then* (1.13) *is not required.*)

Proof. If $a(t) \equiv 0$, the result is obvious from (7.13). Thus we suppose $a(0) > 0$. By Lemma 7.1, the hypothesis implies (7.6).

As in the proof of Lemma 7.3, we obtain the formula (7.13). Integration by parts then yields

$$\Omega(\omega) = \frac{1}{\omega}\int_0^\infty a(t)\sin\omega t\,dt + \frac{q}{\omega}\int_0^\infty a'(t)\sin\omega t\,dt - q\theta. \tag{7.14}$$

To establish (7.8), it suffices since both functions in (7.14) are even, to consider the interval $0 < \omega < \infty$. It is easy to see from (7.14) that if $\sup_{0<\omega<\infty} Z(\omega) = Q < \infty$, where

$$Z(\omega) = \left(\int_0^\infty a(t)\sin \omega t\, dt\right)\left(-\int_0^\infty a'(t)\sin \omega t\, dt\right)^{-1},$$

then (7.8) is satisfied for all $q \geq Q$. We shall show that $Q < \infty$.

From (1.5) we see that $Z(\omega)$ is positive and continuous for $0 < \omega < \infty$. We shall show that $\lim_{\omega \to 0+} Z(\omega)$ and $\lim_{\omega \to \infty} Z(\omega)$ are finite. This then implies $Q < \infty$. One notes that $t\,a(t) \in L_1(0, \infty)$ by Lemma 7.2. Also, $t\,a'(t) \in L_1(0, \infty)$, by (1.5) and because $a(t) \in L_1(0, \infty)$. Hence, by the dominated-convergence theorem,

$$\lim_{\omega \to 0+} \frac{1}{\omega}\int_0^\infty a(t)\sin \omega t\, dt = \int_0^\infty t\,a(t)\,dt,$$

$$\lim_{\omega \to 0+} -\frac{1}{\omega}\int_0^\infty a'(t)\sin \omega t\, dt = -\int_0^\infty t\,a'(t)\,dt = \int_0^\infty a(t)\,dt.$$

Thus, $\lim_{\omega \to 0+} Z(\omega) < \infty$.

We now let $\omega \to \infty$ and investigate the case where $a'(0) = -\infty$. For $n = 1, 2, \cdots$, define

$$\alpha_n(t) = -a''\left(\tfrac{1}{n}\right)\left(t - \tfrac{1}{n}\right) - a'\left(\tfrac{1}{n}\right) \qquad \left(0 \leq t \leq \tfrac{1}{n}\right),$$

$$\alpha_n(t) = -a'(t) \qquad \left(\tfrac{1}{n} \leq t < \infty\right).$$

By (1.5) and (1.13),

$$-\int_0^\infty a'(t)\sin \omega t\, dt \geq \int_0^\infty \alpha_n(t)\sin \omega t\, dt \qquad (0 < \omega < \infty),$$

which together with the definition of $Z(\omega)$ implies

$$Z(\omega) \leq \left(a(0) + \int_0^\infty a'(t)\cos \omega t\, dt\right)\Big/\left(\alpha_n(0) + \int_0^\infty \alpha_n'(t)\cos \omega t\, dt\right).$$

Thus, by the Riemann-Lebesgue lemma,

$$\limsup_{\omega \to \infty} Z(\omega) \leq a(0)/\alpha_n(0).$$

Letting $n \to \infty$, we conclude that $\lim_{\omega \to \infty} Z(\omega) = 0$. (If $a'(0) > -\infty$, the argument is simpler; one can show directly from (1.5), without using (1.13), that

$$\lim_{\omega \to \infty} Z(\omega) = a(0)/(-a'(0)) < \infty.)$$

This completes the proof.

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