THE GENERALIZED BIEBERBACH CONJECTURE FOR SUBORDINATE FUNCTIONS

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1. INTRODUCTION

The Bieberbach conjecture is the assertion that for the class S of functions

$$(1.1) \qquad F(z) = \sum_{n=1}^{\infty} A_n z^n$$

that are regular and schlicht in the unit disk $E(z: |z| < 1)$, the Taylor coefficients satisfy the inequality

$$(1.2) \qquad |A_n| \leq n|A_1| \qquad (n = 2, 3, \cdots).$$

For $n = 2, 3$, and 4, this conjecture is known to be correct for all $F \in S$, and it has been established for all n in the following special cases:

(i) F(z) is real on the real axis [2], [9],

(ii) F(z) maps E onto a domain starlike with respect to the origin [5],

(iii) F(z) maps E onto a spirallike domain [11],

(iv) F(z) maps E onto a domain convex in one direction [7],

(v) F(z) maps E onto a domain D starlike with respect to a point w_0 (in or outside of D); that is, D is composed of segments of straight lines through w_0, with at most one segment on each line [1],

(vi) F(z) is close-to-convex in E [6].

Cases (ii) and (iv) are contained in case (vi), but they were treated earlier. Case (iii), and case (v) when w_0 is outside D, are not included in case (vi).

If the analytic function

$$(1.3) \qquad f(z) = \sum_{n=1}^{\infty} a_n z^n \qquad (|z| < 1)$$

is merely regular (not necessarily schlicht) in E, and in E takes no value omitted by the schlicht function F(z) of (1.1), then we say that f(z) is *subordinate* to F(z) in E, and we write $f(z) \prec F(z)$. In this situation there exists a bounded analytic function $\omega(z)$, regular and satisfying in E the relations

$$\omega(0) = 0, \qquad |\omega(z)| \leq |z| < 1,$$

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for which $f(z) = F(\omega(z))$ [4].

Following W. Rogosinski [10], we call *generalized Bieberbach conjecture* the proposition that when $f(z) \prec F(z)$ and $F(z)$ is schlicht, then

$$|a_n| \leq n|A_1| \qquad (n = 1, 2, \cdots). \tag{1.4}$$

This proposition is known to be true [4] for $n = 1, 2$, and for the two cases (i) and (ii) mentioned above [10]. Since each function $F(z)$ is subordinate to itself, (1.4) implies (1.2).

It is the primary purpose of this note to show that the generalized Bieberbach conjecture is correct for all n, in the more general case (vi). In this situation, there exists a schlicht function

$$\phi(z) = \sum_{n=1}^{\infty} d_n z^n \qquad (|d_1| = 1,\ |z| < 1) \tag{1.5}$$

that maps E onto a convex domain and for which

$$\Re\left\{\frac{F'(z)}{\phi'(z)}\right\} \geq 0 \qquad (z \in E). \tag{1.6}$$

The equality sign in (1.6) can occur only in the trivial case $F(z) = \pm iC\phi(z)$ $(C > 0)$, that is, when $F(z)$ maps E onto a convex domain. For $n > 1$, equality in (1.4) is attained only when $F(z) = \frac{A_1}{\varepsilon} K(\varepsilon z)$, where $|\varepsilon| = 1$ and $K(z) = z(1 - z)^{-2}$.

Even if $F(z)$ is not schlicht, $f(z)$ may still be subordinate to $F(z)$, with $f(z) = F(\omega(z))$. We shall show that in this situation (1.4) holds for a wide class of functions of the form

$$F(z) = \int_0^{2\pi} e^{-it} g(ze^{it})\, d\alpha(t) \qquad (z \in E), \tag{1.7}$$

where $g(z)$ is schlicht and close-to-convex in E, and where $\alpha(t)$ is any nondecreasing function on $[0, 2\pi]$, normalized so that

$$\int_0^{2\pi} d\alpha(t) = 1.$$

We also note that W. Rogosinski [10] has shown that

$$|a_n| \leq |A_1| \qquad (n = 1, 2, \cdots) \tag{1.8}$$

if $f(z) \prec F(z)$, where $F(z)$, given by (1.1), is schlicht and maps E onto a convex domain.

We shall extend (1.8) to the case where $F(z)$ is real on the real axis and maps E onto a domain D convex in the direction of the imaginary axis; that is, where the domain D is comprised of vertical line segments with at most one segment on each vertical line. In such a case, D is obviously symmetric about the real axis, since $F(z)$ is real on the real axis.

The question whether (1.4) is also correct for the cases (iii) and (v), mentioned earlier, remains open.

2. PROOFS OF THEOREMS

THEOREM 1. *Let* $F(z) = \sum_1^\infty A_n z^n$ *be regular and schlicht in* $E(z: |z| < 1)$, *and close-to-convex relative to a schlicht function* $\phi(z) = \sum_1^\infty d_n z^n$ $(|d_1| = 1)$ *that maps* E *onto a convex domain. Let* $f(z) = \sum_s^\infty a_n z^n$ $(a_s \neq 0)$ *be regular and subordinate to* $F(z)$ *in* E. *Then there exists an analytic function* $g(z)$, *regular and subordinate to* $\phi(z)$ *in* E, *for which the function* $f'(z)/g'(z)$ *is regular and has a positive real part in* E. *Furthermore,*

$$|a_n| \leq \left[n - \frac{s(s-1)}{n}\right] \cdot |A_1| \leq n|A_1| \qquad (n \geq s).$$

If $|a_n| = n|A_1|$ *for a particular* $n > 1$, *then*

$$s = 1, \qquad |a_n| = n|A_1| \text{ for all } n, \qquad \text{and } f(z) = F(\eta z),$$

where

$$F(z) = A_1 z(1 - \varepsilon z)^{-2} \qquad \text{and} \qquad |\eta| = |\varepsilon| = 1.$$

Proof. Since $f(z) \prec F(z)$, we can write $f(z) = F(\omega(z))$, where

$$\text{(2.1)} \qquad \omega(z) = \sum_{n=s}^{\infty} \alpha_n z^n, \qquad |\omega(z)| \leq |z| < 1, \qquad \alpha_s \neq 0.$$

Also,

$$\text{(2.2)} \qquad \Re\left\{\frac{F'(z)}{\phi'(z)}\right\} \geq 0 \qquad (|z| < 1).$$

W. Kaplan has shown [3] that $w = F(z)$ maps each circle $|z| = r < 1$ onto a contour with a continuously turning tangent that never turns back on itself through an angle larger than π radians.

Using the function $\omega(z)$ defined in (2.1), we let

$$\text{(2.3)} \qquad g(z) = \phi(\omega(z)) = \sum_{n=s}^{\infty} \beta_n z^n.$$

Since $g(z) \prec \phi(z)$ and $\phi(z)$ maps E onto a convex domain, we have from (1.8) the inequalities

$$\text{(2.4)} \qquad |\beta_n| \leq |d_1| = 1 \qquad (n = s, s+1, \cdots).$$

From the equation $g(z) = \phi(\omega(z))$ we deduce that $\beta_s = d_1 \alpha_s$, so that $|\beta_s| = |\alpha_s|$. Again, since $f(z) = F(\omega(z))$, we see that $a_s = A_1 \alpha_s$, where $A_1 \neq 0$, $a_s \neq 0$. Thus $\alpha_s \neq 0$, $\beta_s \neq 0$, and

$$\left|\frac{a_s}{\beta_s}\right| = \left|\frac{a_s}{\alpha_s}\right| = |A_1| \neq 0.$$

We write $a_s/\beta_s = A_1 e^{i\gamma}$ $(0 \leq \gamma < 2\pi)$. Since

$$f'(z) = F'(\omega(z)) \cdot \omega'(z) \quad \text{and} \quad g'(z) = \phi'(\omega(z)) \cdot \omega'(z),$$

and since F' and ϕ' have no zeros in E, the functions f' and g' have no zeros except the zeros of w'. Consequently, the function

$$\frac{f'(z)}{g'(z)} = \frac{F'(\omega(z))}{\phi'(\omega(z))}$$

is regular in E, and by (2.2) its real part is nonnegative in E. It follows further that the function $P(z)$, defined by the relation

$$\frac{f'(z)}{g'(z)} = |A_1| (i \sin \gamma + \cos \gamma \cdot P(z)) \tag{2.5}$$

if $\cos \gamma > 0$ and by the relation $P(z) = 1$ if $\cos \gamma = 0$, is regular in E, with $P(0) = 1$. Setting $z = 0$ in (2.5), we see that

$$\frac{f'(0)}{g'(0)} = \frac{a_s}{\beta_s} = |A_1| e^{i\gamma}, \qquad \cos \gamma = \frac{1}{|A_1|} \Re \left\{\frac{f'(0)}{g'(0)}\right\} \geq 0,$$

and therefore $\Re P(z) > 0$ in E. Let

$$P(z) = \sum_0^\infty p_n z^n \qquad (p_0 = 1).$$

Since $\Re P(z) > 0$ in E, we have the inequality $|p_n| \leq 2$ for $n = 1, 2, \cdots$. By (2.5),

$$\sum_{n=s}^\infty n a_n z^n = |A_1| \left[e^{i\gamma} + \cos \gamma \cdot \sum_1^\infty p_k z^k \right] \left[\sum_{m=s}^\infty m\beta_m z^m \right],$$

whence $a_s = |A_1| e^{i\gamma} \beta_s$, $|a_s| \leq |A_1|$, and

$$n a_n = |A_1| [n\beta_n e^{i\gamma} + \cos\gamma \cdot \{(n-1)\beta_{n-1} p_1 + (n-2)\beta_{n-2} p_2 + \cdots + s\beta_s p_{n-s}\}].$$

Since $|\beta_m| \leq 1$ for $m = s, s+1, \cdots, n$, and $|p_k| \leq 2$ for $k = 1, 2, \cdots, n-s$, we deduce that for $n \geq s$

$$\begin{aligned} n|a_n| &\leq |A_1| [n + 2\cos \gamma \cdot \{s + (s-1) + \cdots + (n-1)\}] \\ &\leq |A_1| [n + \cos \gamma \cdot \{n(n-1) - s(s-1)\}], \end{aligned}$$

$$|a_n| \leq \left[n - \frac{s(s-1)}{n}\right] \cdot |A_1| \leq n|A_1| \qquad (n \geq s).$$

For $n > 1$, the relation $|a_n| = n|A_1|$ can hold only if $\gamma = 0$, $|p_k| = 2$ for $k = 1, 2, \cdots, n-1$, and $|\beta_m| = 1$ for $m = 1, 2, \cdots, n$. In this case (see [10, p. 70]), $w = \phi(z)$ maps E onto a half-plane and

$$\phi(z) = \frac{d_1 z}{1 - \varepsilon z}, \qquad |d_1| = |\varepsilon| = 1.$$

Since

$$\frac{\varepsilon}{d_1} g(z) \prec \frac{\varepsilon}{d_1} \phi(z) \prec \frac{z}{1 - z},$$

we have the inequality

$$\Re\left\{\frac{2\varepsilon}{d_1} g(z) + 1\right\} > 0 \qquad (|z| < 1).$$

Therefore $|\beta_1| = 1$ only if $g(z)$ has the form given by

$$\frac{2\varepsilon}{d_1} g(z) + 1 = \frac{1 + \eta z}{1 - \eta z} \qquad (|\eta| = 1),$$

that is,

$$\frac{\varepsilon}{d_1} g(z) = \frac{\eta z}{1 - \eta z}.$$

Furthermore, when $n > 1$ we must have $|p_1| = 2$ in order to obtain $|a_n| = n\,|A_1|$. Consequently,

$$P(z) = \frac{1 + \varepsilon z}{1 - \varepsilon z}, \qquad (|\varepsilon| = 1).$$

However, $|a_n| = n\,|A_1|$ only when $\beta_n, \beta_{n-1} p_1, \cdots, \beta_1 p_{n-1}$ all have the same amplitude. Hence $\eta = \varepsilon$. In this case $g'(z) = d_1(1 - \varepsilon z)^{-2}$.

For $n > 1$ and $|a_n| = n\,|A_1|$, (2.5) reduces to

$$\frac{f'(z)}{g'(z)} = |A_1|\, P(z), \qquad f'(z) = \frac{|A_1|\, d_1 (1 + \varepsilon z)}{(1 - \varepsilon z)^3},$$

$$f(z) = a_1 z (1 - \varepsilon z)^{-2}, \qquad a_1 = |A_1|\, d_1, \qquad |a_1| = |A_1|.$$

The conditions $|a_1| = |A_1|$ and $f \prec F$ imply that $F(z) = f(\bar{\eta} z)$ for some η ($|\eta| = 1$). Thus $|a_n| = n\,|A_1|$ for some $n > 1$ only if $F(z)$ is of the form $A_1 z(1 - e^{i\alpha} z)^{-2}$ (α real), in which case $|a_n| = n\,|A_1|$ for all n and $f(z) = F(\eta z)$, where $|\eta| = 1$. This completes the proof of Theorem 1.

The following is an extension of Theorem 1.

THEOREM 2. *Let* $\mu(z) = \sum_1^\infty c_n z^n$ *be regular, schlicht, and close-to-convex in* $E(z\colon |z| < 1)$. *Let* $\alpha(t)$ *be a real function, nondecreasing in* $[0, 2\pi]$, *and normalized so that* $\int_0^{2\pi} d\alpha(t) = 1$. *Let*

$$F(z) = \sum_1^\infty A_n z^n = \int_0^{2\pi} e^{-it} \mu(e^{it} z)\, d\alpha(t), \qquad A_1 = c_1 \neq 0.$$

Let $f(z) = \sum_1^\infty a_n z^n \prec F(z)$ *in* E. *Then* $|a_n| \leq n\,|A_1|$ $(n = 1, 2, \cdots)$.

Proof. $f(z) = F(\omega(z))$, where

$$\omega(z) = \sum_1^\infty \alpha_n z^n \quad \text{and} \quad |\omega(z)| \leq |z| < 1.$$

If we write

$$[\omega(z)]^k = \sum_{n=k}^\infty \alpha_n^{(k)} z^n,$$

then

$$f(z) = \sum_{k=1}^\infty A_k[\omega(z)]^k, \qquad a_n = \sum_{k=1}^n \alpha_n^{(k)} A_k.$$

Let $h(z) = \mu(e^{it}\omega(z)) = \sum_1^\infty b_n z^n$. Then

$$h(z) \prec \mu(e^{it}z) = \sum_1^\infty c_n e^{int} z^n, \qquad b_n = \sum_{k=1}^n \alpha_n^{(k)} (c_k e^{ikt}).$$

Since $h(z) \prec \mu(z)$ and $\mu(z)$ is close-to-convex in E, it follows from Theorem 1 that $|b_n| \leq n\,|c_1|$. Hence

$$\left| \sum_{k=1}^n \alpha_n^{(k)} c_k e^{ikt} \right| \leq n\,|c_1| = n\,|A_1|,$$

$$A_k = c_k \int_0^{2\pi} e^{(k-1)it}\, d\alpha(t),$$

$$a_n = \sum_{k=1}^n \alpha_n^{(k)} A_k = \int_0^{2\pi} \left[\sum_{k=1}^n \alpha_n^{(k)} c_k e^{ikt} \right] e^{-it}\, d\alpha(t),$$

$$|a_n| \leq \int_0^{2\pi} \left| \sum_{k=1}^n \alpha_n^{(k)} c_k e^{ikt} \right| d\alpha(t) \leq n\,|A_1| \int_0^{2\pi} d\alpha(t) = n\,|A_1|.$$

This completes the proof of Theorem 2.

We shall now extend Rogosinski's result.

THEOREM 3. *Let* $F(z) = \sum_1^\infty A_n z^n$ *be regular in* E *and real on the real axis, and let it map* E *onto a domain* D *convex in the direction of the imaginary axis. Let* $f(z) = \sum_1^\infty a_n z^n$ *be regular and subordinate to* $F(z)$ *in* E. *Then*

$$|a_n| \leq |A_1| \qquad (n = 1, 2, \cdots).$$

Proof. $F(z)$ has the representation (see [7])

$$F(z) = A_1 \int_0^{\pi} \int_0^{z} \frac{du}{1 - 2u \cos t + u^2}\, d\alpha(t),$$

where $\alpha(t)$ is nondecreasing on $[0, \pi]$ and $\int_0^{\pi} d\alpha(t) = 1$. Thus

$$A_k = \frac{A_1}{k} \int_0^{\pi} \frac{\sin kt}{\sin t}\, d\alpha(t).$$

Let $f(z) = F(\omega(z))$, and define

$$h(z) = A_1 \int_0^{\omega(z)} \frac{du}{1 - 2u \cos t + u^2} = \sum_1^{\infty} b_n z^n.$$

For each t, the function

$$C(z) = A_1 \int_0^{z} \frac{du}{1 - 2u \cos t + u^2} = A_1 z + \cdots$$

maps E onto a convex domain, since the function

$$z C'(z) = A_1 z(1 - 2z \cos t + z^2)^{-1}$$

is starlike in E. Also, $h(z) \prec C(z)$, from which it follows that $|b_n| \leq |A_1|$. However,

$$b_n = \sum_{k=1}^{n} \alpha_n^{(k)} \cdot \frac{A_1}{k} \cdot \frac{\sin kt}{\sin t},$$

where

$$[\omega(z)]^k = \sum_{n=k}^{\infty} \alpha_n^{(k)} z^n.$$

Hence

$$\left| A_1 \sum_{k=1}^{n} \frac{\alpha_n^{(k)}}{k} \frac{\sin kt}{\sin t} \right| = |b_n| \leq |A_1|,$$

$$a_n = \sum_{k=1}^{n} \alpha_n^{(k)} A_k = \int_0^{\pi} \left[A_1 \sum_{k=1}^{n} \frac{\alpha_n^{(k)}}{k} \frac{\sin kt}{\sin t} \right] d\alpha(t),$$

$$|a_n| \leq \int_0^{\pi} \left| A_1 \sum_{k=1}^{n} \frac{\alpha_n^{(k)}}{k} \frac{\sin kt}{\sin t} \right| d\alpha(t) \leq |A_1| \cdot \int_0^{\pi} d\alpha(t) = |A_1|.$$

This completes the proof of Theorem 3.

THEOREM 4. *Let* $G(z) = \sum_0^\infty A_n z^n$ *be regular in* $E(z: |z| < 1)$. *Let* $\Re(zG(z))' > 0$ *in* E, *so that* $zG(z)$ *is univalent in* E. *Let*

$$f(z) = A_0 + \sum_1^\infty a_n z^n \prec G(z)$$

in E. *Then* $|a_n| \leq \Re A_0$ $(n = 1, 2, \cdots)$. *The equality sign holds if* $f(z) = G(z^n)$, *where*

$$G(z) = A_0 + 2(\Re A_0)\left(-\frac{1}{z} - \frac{1}{z^2}\log(1 - z)\right).$$

Proof. Let $G(z) + zG'(z) = i|A_0|\sin\alpha + |A_0|\cos\alpha \cdot P(z)$, where $A_0 = |A_0|e^{i\alpha}$. Since $\Re[G(z) + zG'(z)] > 0$ in E, we see (by setting $z = 0$) that $\Re A_0 > 0$ and $\cos\alpha > 0$. It follows that $P(z)$ is regular and $\Re P(z) > 0$ in E, with $P(0) = 1$. From the representation

$$P(z) = \int_0^{2\pi} \frac{1 + ze^{it}}{1 - ze^{it}}\, d\alpha(t) \qquad \left(\int_0^{2\pi} d\alpha(t) = 1, \quad \alpha(t) \uparrow \text{ in } [0, 2\pi]\right)$$

we obtain the formula

$$(k + 1)A_k = 2|A_0|\cos\alpha \cdot \int_0^{2\pi} e^{kit}\, d\alpha(t) \qquad (k = 1, 2, \cdots).$$

Since $f(z) - A_0 \prec G(z) - A_0$, we have the relations $f(z) - A_0 = G(\omega(z)) - A_0$ and

$$a_n = \sum_{k=1}^{n} \alpha_n^{(k)} A_k = 2|A_0|\cos\alpha \cdot \int_0^{2\pi}\left[\sum_{k=1}^{n} \alpha_n^{(k)} \frac{e^{kit}}{k + 1}\right] d\alpha(t),$$

where

$$[\omega(z)]^k = \sum_{n=k}^{\infty} \alpha_n^{(k)} z^n \qquad (k = 1, 2, \cdots).$$

It is known [8] that the function

$$w = -1 - \frac{2}{z}\log(1 - z) = 1 + 2\sum_{k=1}^{\infty} \frac{z^k}{k + 1}$$

maps E onto a convex domain. Consequently, if we define

$$h(z) = \sum_{k=1}^{\infty} \frac{e^{kit} z^k}{k + 1} \qquad (0 \leq t \leq 2\pi)$$

and determine $g(z)$ by the equation

$$\sum_{1}^{\infty} b_n(t) z^n = g(z) = h(\omega(z)) \prec h(z),$$

it follows from (1.8) that $|b_n(t)| \leq |h'(0)| = 1/2$. However,

$$b_n(t) = \sum_{k=1}^{n} \alpha_n^{(k)} \left(\frac{e^{kit}}{k+1}\right).$$

Therefore

$$|a_n| \leq 2|A_0| \cos \alpha \cdot \int_0^{2\pi} |b_n(t)|\, d\alpha(t) \leq \Re A_0,$$

which was to be proved.

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