LINEAR PERTURBATIONS OF CONNEXIONS

N. Hicks

This study was motivated by the following natural question. If one takes the second fundamental form on a convex surface to be a new Riemannian metric, then how are the new connexion and curvature related to the usual ones? The question leads to the definition of a natural connexion associated with a smooth nonsingular linear-transformation-valued tensor on a Riemannian manifold, and to related invariants. In particular, the relation between all semi-Riemannian structures and one fixed Riemannian structure can be studied. Our methods are applied to the Weingarten map of a hypersurface in a Riemannian manifold, and to the natural linear operator associated with a vector field on a Riemannian manifold.. The meaning of the invariants in the latter case is not yet clear.

All terms that are not explicitly defined can be found in reference [1].

1. THE ASSOCIATED CONNEXIONS

Let M be an n-dimensional C^∞ manifold with connexion D. Let L be a function that assigns to each p in M a linear map L_p on the tangent space at p, M_p. Let L be C^∞ in the sense that for each C^∞ vector field X with domain U, the field $L(X)_p = L_p(X_p)$ is C^∞ on U. Let L be nonsingular on each tangent space, and define two connexions D" and D' by

$$D''_X(Y) = L^{-1} D_X L(Y), \tag{1}$$

$$D'_X(Y) = \frac{1}{2}[D_X Y + D''_X Y], \tag{2}$$

where X and Y are smooth (C^∞) vector fields on M. One verifies easily that D" and D' satisfy the conditions of a covariant differentiation operator. Recall that for any connexion D, the torsion Tor_D and curvature R_D are defined by

$$\mathrm{Tor}_D(X, Y) = D_X Y - D_Y X - [X, Y], \tag{3}$$

$$R_D(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \tag{4}$$

where X, Y, and Z are smooth vector fields on M. For each smooth, linear-map-valued tensor L, we define

$$\mathrm{Tor}_L(X, Y) = D_X LY - D_Y LX - L[X, Y], \tag{5}$$

$$R_L(X, Y)Z = D_X L D_Y Z - D_Y L D_X Z - D_{[X,Y]} LZ, \tag{6}$$

where Tor_L is always a tensor but R_L is not necessarily a tensor (see Theorem 1). Notice that Tor_L and R_L are really functions depending upon L and D. The following proposition follows easily from the definitions.

Received February 9, 1965.
This work was supported by NSF grant GP 1815.

PROPOSITION 1. *Letting* $\mathrm{Tor} = \mathrm{Tor}_D$, $R = R_D$, $\mathrm{Tor}' = \mathrm{Tor}_{D'}$, $R' = R_{D'}$, $\mathrm{Tor}'' = \mathrm{Tor}_{D''}$, *and* $R'' = R_{D''}$, *one gets*

$$\mathrm{Tor}''(X, Y) = L^{-1}\mathrm{Tor}_L(X, Y), \tag{7}$$

$$R''(X, Y)Z = L^{-1}R(X, Y)LZ, \tag{8}$$

$$\mathrm{Tor}'(X, Y) = \frac{1}{2}[\mathrm{Tor}(X, Y) + \mathrm{Tor}''(X, Y)], \tag{9}$$

$$R'(X, Y)Z = \frac{1}{4}[R(X, Y)Z + R''(X, Y)Z + L^{-1}R_L(X, Y)Z + R_{L^{-1}}(X, Y)LZ]. \tag{10}$$

In particular, this proposition implies that

$$S_L(X, Y)Z = \frac{1}{2}[L^{-1}R_L(X, Y)Z + R_{L^{-1}}(X, Y)LZ] \tag{11}$$

is a tensor.

THEOREM 1. *The following assertions are equivalent:*

(a) $D = D''$, (b) $D = D'$, (c) $D' = D''$, (d) $LD_X = D_X L$ *for all fields* X,

(e) $\Delta L = 0$, *where* Δ *is the general covariant derivative associated with* D,

(f) R_L *is a tensor.*

Proof. The first four are trivial. If we consider L as a tensor of type 1, 1, then ΔL is the tensor of type 1, 2 with

$$(\Delta L)(\omega, X, Y) = (D_Y L)(\omega, X) = Y\omega(LX) - (D_Y\omega)(LX) - \omega(LD_Y X)$$
$$= Y\omega(LX) - Y\omega(LX) + \omega(D_Y LX) - \omega(LD_Y X) = \omega(D_Y LX - LD_Y X),$$

where ω is a C^∞ 1-form on M. Thus $\Delta L = 0$ if and only if $D_Y L = LD_Y$ for all Y.

If (d) holds, then $R_L(X, Y)Z = LR(X, Y)Z$ is a tensor. Let f be a C^∞ real-valued function on M, and notice that

$$R_L(X, fY)Z = fR_L(X, Y)Z + (Xf)[LD_Y Z - D_Y LZ].$$

Thus, if R_L is a tensor, one may choose X and f on a neighborhood with $Xf = 1$, so that $LD_Y = D_Y L$, which proves (d).

For the rest of this paper, suppose M is Riemannian with metric tensor $\langle\ ,\ \rangle$ and Riemannian connexion D.

THEOREM 2. *Let* $\langle X, Y\rangle'' = \langle LX, LY\rangle$ *define a new Riemannian metric on* M. *Then* D'' *is* $\langle\ ,\ \rangle''$ *metric-preserving, and* D'' *is the Riemannian metric associated with* $\langle\ ,\ \rangle''$ *if and only if* $\mathrm{Tor}_L = 0$.

Proof. For fields X, Y, and Z,

$$\langle D''_X Y, Z\rangle'' + \langle Y, D''_X Z\rangle'' = \langle D_X LY, LZ\rangle + \langle LY, D_X LZ\rangle$$
$$= X\langle LY, LZ\rangle = X\langle Y, Z\rangle'',$$

since D is Riemannian for $\langle\ ,\ \rangle$.

By Proposition 1, Tor" = 0 if and only if $\mathrm{Tor}_L = 0$.

THEOREM 3. *If* L *is self-adjoint (symmetric) with respect to* $\langle\ ,\ \rangle$, *define the semi-Riemannian metric* $\langle X, Y\rangle' = \langle X, LY\rangle = \langle LX, Y\rangle$. *The connexion* D' *is* $\langle\ ,\ \rangle'$ *metric-preserving, and* D' *is the Riemannian connexion associated with* $\langle\ ,\ \rangle'$ *if and only if* $\mathrm{Tor}_L = 0$.

Proof. Clearly,

$$\begin{aligned}\langle D'_X Y, Z\rangle' &+ \langle Y, D'_X Z\rangle' \\ &= \tfrac{1}{2}[\langle D_X Y + L^{-1} D_X LY, LZ\rangle + \langle LY, D_X Z + L^{-1} D_X LZ\rangle] \\ &= \tfrac{1}{2}[\langle D_X Y, LZ\rangle + \langle Y, D_X LZ\rangle + \langle D_X LY, Z\rangle + \langle LY, D_X Z\rangle] \\ &= X\langle Y, Z\rangle'.\end{aligned}$$

Thus D' preserves $\langle\ ,\ \rangle'$. Since Tor = 0,

$$\mathrm{Tor}' = 0 \Leftrightarrow \mathrm{Tor}'' = 0 \Leftrightarrow \mathrm{Tor}_L = 0.$$

We can now iterate the above procedure for defining D' and D" in terms of D to obtain the semi-Riemannian connexion associated with the metric tensors $\langle L^r X, L^r Y\rangle$ (for arbitrary nonsingular L) and $\langle L^r X, Y\rangle$ (for symmetric nonsingular L) for each integer $r > 0$. In addition, the one-to-one correspondence between semi-Riemannian metrics and nonsingular self-adjoint maps shows that each semi-Riemannian connexion D' is related to D by equation (2).

We shall now use the above analysis to relate various "plane" curvatures (Riemannian curvatures of two-dimensional subspaces of a tangent space).

PROPOSITION 2. *Let* B *be a* C^∞ *real-valued tensor on* M *of type* 0, 4 *such that for all* X *and* Y *in* M_p

$$B(X, Y, X, Y) = -B(Y, X, X, Y) = -B(X, Y, Y, X). \tag{12}$$

If P *is any two-dimensional subspace of* M_p *and* X, Y *a base for* P, *then the plane curvature of* P *relative to* B *and the metric tensor* $\langle\ ,\ \rangle$ *defined by*

$$K(P) = B(X, Y, X, Y)/A(X, Y) \tag{13}$$

is independent of the base X, Y, *where* $A(X, Y) = \langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2$.

The proof is a simple exercise involving the use of (12). Now define

$$K(X, Y, Z, W) = \langle X, R(Z, W)Y\rangle, \tag{14}$$

$$K'(X, Y, Z, W) = \langle X, R'(Z, W)Y\rangle', \tag{15}$$

$$\begin{cases} Q(X, Y, Z, W) = \tfrac{1}{4}[\langle LX, R(Z, W)Y\rangle + \langle X, R(LZ, W)Y\rangle \\ \qquad + \langle X, R(Z, LW)Y\rangle + \langle X, R(Z, W)LY\rangle], \end{cases} \tag{16}$$

(17) $$S(X, Y, Z, W) = \langle LX, S_L(Z, W)Y \rangle .$$

PROPOSITION 3. *The tensors* K *and* Q *each satisfy* (12). *If* L *is symmetric, then* K' *and* S *satisfy* (12).

Proof. That K and K' satisfy (12) is clear, since they are the classical Riemann-Christoffel tensors of type 0, 4. Then Q satisfies (12), since K does. Actually K, K', and Q each satisfy the equation

(18) $$B(X, Y, Z, W) = B(Z, W, X, Y),$$

but S may not. To show that S satisfies (12), notice that

$$R(X, Y)\, LY + R(LY, X)Y + R(Y, LY)X = 0$$

by the first Bianchi identify for the connexion D. Hence

$$\langle R(X, Y)LY, X \rangle = \langle R(X, LY)Y, X \rangle$$

and

$$Q(X, Y, X, Y) = \frac{1}{2}[\langle LX, R(X, Y)Y \rangle + \langle X, R(X, Y)LY \rangle].$$

Then, by Proposition 1,

$$K'(X, Y, X, Y) = \langle X, R'(X, Y)Y \rangle '$$

(19) $$= \langle LX, \frac{1}{4}R(X, Y)Y + \frac{1}{4}L^{-1}R(X, Y)LY + \frac{1}{2}S_L(X, Y)Y \rangle$$

$$= \frac{1}{2}Q(X, Y, X, Y) + \frac{1}{2}S(X, Y, X, Y),$$

which shows that S satisfies (12), since K' and Q satisfy (12).

Combining Propositions 2 and 3, one defines the plane curvatures $K'(P)$, $K_Q(P)$, and $K_S(P)$ belonging to a nonsingular self-adjoint smooth L by

$$K'(P) = K'(X, Y, X, Y)/A'(X, Y), \qquad K_Q(P) = Q(X, Y, X, Y)/A'(X, Y),$$

$$K_S(P) = S(X, Y, X, Y)/A'(X, Y),$$

where X and Y are a base for P.

THEOREM 4. *For each plane section* P,

(20) $$K'(P) = \frac{1}{2}[K_Q(P) + K_S(P)].$$

Proof. This follows from (19).

Theorem 4 gives a natural decomposition of the (semi-) Riemannian curvature $K'(P)$ of the metric in Theorem 3 when $Tor_L = 0$. It is also true that both R_L and S_L satisfy the first Bianchi identity if $Tor_L = 0$.

2. APPLICATIONS AND EXAMPLES

First consider the situation where L is the Weingartern map for a hypersurface M of a flat Riemannian manifold $\overline{M}$ with metric tensor $\langle\ ,\ \rangle$ and Riemannian connexion $\overline{D}$. If we let N be a unit C^∞ normal field on M (which always exists locally), then $L(X) = \overline{D}_X N$ for X in M_p (see reference [1]). Since $\overline{M}$ is flat, the Codazzi-Mainardi equation implies $\mathrm{Tor}_L = 0$, and the Gauss curvature equation becomes

$$R(X, Y)Z = \langle LY, Z\rangle L(X) - \langle LX, Z\rangle L(Y) \tag{21}$$

for X, Y, and Z tangent to M.

For C^∞ fields X and Y tangent to M, we let

$$A_L(X, Y) = LD_X Y - D_X LY, \tag{22}$$

and we verify easily that A_L is linear over the module of C^∞ vector fields with C^∞ functions as coefficients, which implies that A_L is a vector-valued tensor. Moreover, since $\mathrm{Tor}_D = 0$ and $\mathrm{Tor}_L = 0$, we compute

$$A_L(X, Y) = L(D_Y X + [X, Y]) - (D_Y LX + L[X, Y]) = A_L(Y, X).$$

THEOREM 5. *Let* M *be a hypersurface of a flat Riemannian manifold* $\overline{M}$ *with unit normal* N *on* M, *and suppose the total imbedded curvature* (det L) *of* M *in* $\overline{M}$ *is not zero at* m *in* M. *Let* P *be any plane section of* M_m *with orthonormal base* X, Y. *Then*

$$K'(P) = K_Q(P) + K_A(P), \tag{23}$$

where

$$K_A(P) = \langle A_{L^{-1}}(X, A_L(Y, Y)) - A_{L^{-1}}(Y, A_L(X, Y)), LX\rangle / 4A'(X, Y) \tag{24}$$

and

$$K_Q(P) = \frac{1}{2}(\langle LX, X\rangle + \langle LY, Y\rangle). \tag{25}$$

Proof. Using the Gauss curvature equation, one computes

$$\begin{aligned} 2Q(X, Y, Z, W) = {} & \langle X, L^2 Z\rangle \langle Y, LW\rangle - \langle X, L^2 W\rangle \langle Y, LZ\rangle \\ & + \langle X, LZ\rangle \langle Y, L^2 W\rangle - \langle X, LW\rangle \langle Y, L^2 Z\rangle \end{aligned}$$

(this relation holds for every Weingarten map L). To compute $S_L(Z, W)Y$, one notes that

$$\begin{aligned} R_L(Z, W)Y &= D_Z LD_W Y - D_W LD_Z Y - D_{[Z,W]} LY \\ &= D_Z LD_W Y - D_W LD_Z Y + R(Z, W)LY - D_Z D_W LY + D_W D_Z LY \\ &= D_Z A_L(W, Y) - D_W A_L(Z, Y) + \langle L^2 W, Y\rangle L(Z) - \langle L^2 Z, Y\rangle LW. \end{aligned}$$

A similar computation for $R_{L^{-1}}(Z, W)LY$ yields the formula

$$S_L(Z, W)Y = \frac{1}{2}[L^{-1} R_L(Z, W)Y + R_{L^{-1}}(Z, W)LY]$$

$$= \frac{1}{2}[\langle L^2 W, Y\rangle Z - \langle L^2 Z, Y\rangle W + \langle LW, Y\rangle L(Z) - \langle LZ, Y\rangle L(W)$$

$$+ A_{L^{-1}}(Z, A_L(W, Y)) - A_{L^{-1}}(W, A_L(Z, Y))].$$

Thus

$$S(X, Y, Z, W) = Q(X, Y, Z, W) + \frac{1}{2}\langle A_{L^{-1}}(Z, A_L(W, Y)) - A_{L^{-1}}(W, A_L(Z, Y)), LX\rangle,$$

and (24) follows from this and Theorem 4.

For (25), we use the relation

$$2Q(X, Y, X, Y) = \langle L^2X, X\rangle \langle LY, Y\rangle + \langle L^2Y, Y\rangle \langle LX, X\rangle - 2\langle L^2X, Y\rangle \langle LX, Y\rangle$$

and the fact that $A'(X, Y) \neq 0$ since L_m is nonsingular.

COROLLARY. *If* $\overline{M} = R^3$ *and* $K(m) \neq 0$, *then* $K_Q(M_m) = H(m)/2$ *and*

$$K' = H/2 + K_A, \tag{26}$$

where $P = M_m$ *and* $H(m)$ *is the mean curvature of* M.

In one sense this corollary solves the original problem posed in the introduction, for it relates the (semi-) Riemannian curvature K' of the nonsingular metric $\langle X, Y\rangle' = \langle LX, Y\rangle$ to the mean curvature. The nature of the invariant K_A is not yet clear; however, one can obtain an expression for K_A in the neighborhood of a non-umbilic point.

THEOREM 6. *Let* M *be a smooth surface in* R^3 *with* $K(m) \neq 0$ *at a non-umbilic point* m *in* M. *Let* h *and* k *be the principal curvatures of* M *on a non-umbilic neighborhood* U *of* m *with* $h > k$ *and* $K \neq 0$ *on* U. *Let* X *and* Y *be* C^∞ *unit-orthogonal principal fields on* U *with* $LY = hY$ *and* $LX = kX$. *Then*

$$K' = \frac{H}{2} + \frac{1}{4K}\left[(Xh)X\left(\log\frac{h}{k}\right) + (Yk)Y\left(\log\frac{k}{h}\right)\right] \tag{27}$$

on U.

Proof. The existence of U, X, Y and the C^∞ nature of h and k on U are shown in [1, Chapter 3, Section 1] where one also finds the formulas

$$D_X Y = aX, \quad D_Y X = bY, \quad D_X X = -aY, \quad D_Y Y = -bX \tag{28}$$

with

$$a = (Yk)/(h-k) \quad \text{and} \quad b = -(Xh)/(h-k). \tag{29}$$

Using (28), (29), and the relations $LY = hY$ and $LX = kX$, one computes:

$$A_L(Y, Y) = b(h-k)X - (Yh)Y, \tag{30}$$

$$A_L(X, Y) = -a(h-k)X - (Xh)Y, \tag{31}$$

(32) $$\langle A_{L^{-1}}(X, A_L(Y, Y)), LX \rangle = -(Yh)(Yk)/h - (Xh)(Xk)/k,$$

(33) $$\langle A_{L^{-1}}(Y, A_L(X, Y)), LX \rangle = -(Yk)^2/k - (Xh)^2/h.$$

For example, (30) follows from the equations

$$A_L(Y, Y) = LD_Y Y - D_Y LY = L(-bX) - D_Y(hY) = -bkX - (Yh)Y + hbX.$$

Substitution of (32) and (33) into (23) yields (27).

COROLLARY 1. *Let* M *be a smooth surface in* R^3 *with constant mean curvature and nonvanishing Gauss curvature on a non-umbilic neighborhood* U. *Then*

(34) $$K' = H\left[\frac{1}{2} + \frac{1}{4K^2}\langle \text{grad } h, \text{grad } h\rangle\right]$$

on U.

Proof. Since U is non-umbilic, h and k ($h > k$) are C^∞ on U. Since $H = h + k$ is constant, $(Zh) = -(Zk)$ for all vectors Z tangent to U. Hence by (27)

$$K' = (H/2) + (H/4K)[(Yh)^2 + (Xh)^2],$$

and (34) follows.

COROLLARY 2. *On a non-umbilic neighborhood of a minimal surface* $K' = 0$, *thus the second fundamental form defines a semi-Riemannian metric with zero curvature.*

One might suspect that the condition $D' = D$ on a surface M in R^3 is necessary and sufficient for M to be a sphere, and this is indeed the case.

THEOREM 7. *If the connexion induced on a complete connected surface by its second fundamental form is the usual Riemannian connexion, then the surface is a sphere.*

Proof. The condition $D = D'$ implies that $R(X, Y)Z = R'(X, Y)Z$, hence

$$K' = \langle R(X, Y)Y, X\rangle'/A'(X, Y) = k = \langle R(Y, X)X, Y\rangle'/A'(X, Y) = h,$$

where X and Y are orthonormal principal vectors at any point with $L(X) = kX$ and $L(Y) = h(Y)$. Thus $k = h$, and all points are umbilics. Since L must be nonsingular for D' to be globally defined, the surface is a sphere.

For the second application of the theory, let M be a Riemannian manifold, and let T be a C^∞ vector field on M. Define the linear-map-valued tensor A on M by

(35) $$A_m(X) = D_X T \quad (\text{all } X \text{ in } M_m),$$

where D is the Riemannian connexion. Since D is C^∞, A is C^∞. Let G(T) be the natural 1-form on M associated with T *via* the metric; that is, let

$$G(T)(X) = \langle T, X\rangle.$$

Then T is *closed* by definition if $dG(T) = 0$, or equivalently, if A is self-adjoint (see [2]). Whenever A is nonsingular, one can define the associated connexions D' and D". In addition, if A is symmetric and $\mathrm{Tor}_A = 0$, then the plane curvatures K', K_Q, and K_S are defined.

To begin a study of these associated curvatures, let $M = R^2$ with its usual metric and connexion D. The following example shows that K_S and K' are not always trivial.

Example. Let $\phi = xy^2 + x^2$ and $T = \mathrm{grad}\,\phi = (y^2 + 2x, 2xy)$. If $X = (g_1, g_2)$ is a vector on R^2, then

$$A(X) = D_X T = (2g_1 + 2y g_2, 2y g_1 + 2x g_2). \tag{36}$$

Thus A is nonsingular if $4x - 4y^2 \neq 0$; therefore let M be R^2 minus the parabola where $x = y^2$. Let $X = (1, 0)$ and $Y = (0, 1)$ be fields on M. Then $R_A(X, Y)Y = 0$, since $D_Y Y = D_X Y = [X, Y] = 0$. But

$$A^{-1}(g_1, g_2) = \left(\frac{xg_1 - yg_2}{2x - 2y^2}, \frac{g_2 - yg_1}{2x - 2y^2}\right), \tag{37}$$

$$A(Y) = (2y, 2x), \quad D_Y AY = (2, 0), \quad D_X AY = (0, 2),$$

and

$$R_{A^{-1}}(X, Y)AY = (x, -y)/(x - y^2)^2, \tag{38}$$

while

$$A'(X, Y) = 2(2x) - (2y)^2 = 4x - 4y^2. \tag{39}$$

Hence

$$K_S = 1/4(x - y^2)^2. \tag{40}$$

Since $R = 0$ on M, $Q = 0$ and $K_Q = 0$. Thus $K' = 1/8(x - y^2)^2$.

We conclude this study with a theorem that identifies the closed vector fields T on R^2 for which D' = D.

THEOREM 8. *A field* T *on* R^2 *induces a symmetric nonsingular map* A *with* D' = D *if and only if* T *is of the form* $(ax + by + c, bx + dy + e)$, *where* a, b, c, d, *and* e *are real constants with* $ad - b^2 \neq 0$.

Proof. Let $T = (f_1, f_2)$, where the f_i are C^∞ real-valued functions on R^2. Let $X = (g_1, g_2)$ and $Y = (a_1, a_2)$, where the g_i are C^∞. By Theorem 1, D' = D if and only if $D_Y A(X) = A(D_Y X)$ for all X and Y. Letting $g_x = \partial g/\partial x$, $g_y = \partial g/\partial y$, and taking $X = (1, 0)$, we see that $A(X) = ((f_1)_x, (f_2)_x)$ and

$$D_Y A(X) = (a_1(f_1)_{xx} + a_2(f_1)_{xy}, a_1(f_2)_{xx} + a_2(f_2)_{xy}).$$

But $D_Y X = 0$, hence $A(D_Y X) = 0$. Since Y is arbitrary,

$$(f_1)_{xx} = (f_1)_{xy} = (f_2)_{xx} = (f_2)_{xy} = 0.$$

Similarly, if $X = (0, 1)$, then $(f_1)_{yy} = (f_2)_{yy} = 0$. Thus f_1 and f_2 are of the linear form stated in the theorem with the necessary conditions on the constants to assure that A is symmetric and nonsingular.

REFERENCES

1. N. Hicks, *Notes on differential geometry,* Van Nostrand, Princeton, 1965.

2. ———, *Closed vector fields,* Pacific J. Math. 15 (1965), 141-151.

The University of Michigan