ON THE UNIVALENCE OF A CERTAIN INTEGRAL

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In a recent paper, P. L. Duren, H. S. Shapiro, and A. L. Shields [1] showed that if f belongs to the class S of functions that are regular and univalent in the unit disk D, and if $0 \leq |\alpha| \leq (\sqrt{5} - 2)/3$, then the function

$$f_\alpha(z) = \int_0^z [f'(t)]^\alpha \, dt$$

also belongs to S. They knew that the Koebe function $K(z) = z/(1 - z)^2$ loses its univalency under some transformations $K \to K_\alpha$ with $|\alpha| < 1$ (private communication), but they had no example of a function f in S for which some f_α $(0 < \alpha < 1)$ is not univalent. We shall now identify a subclass of functions f in S for which f_α is univalent whenever $0 < \alpha < 1$. On the other hand, corresponding to each value α $(|\alpha| > 1/3,\ \alpha \neq 1)$ we shall exhibit a function f in S whose transform f_α is not univalent.

THEOREM 1. *If* f *belongs to* S *and is close-to-convex, then* f_α *belongs to* S *and is close-to-convex, whenever* $0 \leq \alpha \leq 1$.

This proposition was proved independently by M. O. Reade and P. L. Duren (private communications). Their proofs are similar and have the advantage of being complete within themselves, whereas the author's original proof employed a strong result of W. Kaplan [2, Theorem 2]. The proof of Reade and Duren is as follows: by definition (see [2]), f is close-to-convex if and only if $f'(z) = p(z)\phi'(z)$, where p is a function with positive real part in D and ϕ is convex in D. The relation $f' = p\phi'$ implies

$$f'_\alpha = (f')^\alpha = p^\alpha (\phi')^\alpha = p^\alpha \phi'_\alpha .$$

Since p^α has positive real part, it remains to show that ϕ_α is convex. The condition for convexity of ϕ is

$$\Re\left(z \frac{\phi''}{\phi'}\right) > -1 \qquad (z \in D).$$

It follows from the relation $z\phi''_\alpha/\phi'_\alpha = \alpha z \phi''/\phi'$ that the transformation $\phi \to \phi_\alpha$ preserves convexity for $0 \leq \alpha \leq 1$. This concludes the proof of Theorem 1.

THEOREM 2. *Corresponding to each complex number* α $(|\alpha| > 1/3,\ \alpha \neq 1)$, *the class* S *contains a function* f *of the form*

$$f(z) = \exp[\mu \log(1 - z)] \tag{1}$$

such that $f_\alpha \notin S$.

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The choice of the branch in (1) is immaterial, because it does not affect the validity of the following auxiliary proposition.

LEMMA. *The function* (1) *is univalent in* D *if and only if* μ *lies in one of the closed disks*

$$|\mu + 1| \leq 1, \qquad |\mu - 1| \leq 1. \tag{2}$$

Proof. If $\mu = \rho e^{i\phi}$ (ϕ real, $\rho > 0$), then the function $g(z) = \mu \log(1 - z)$ maps D onto a subset B of a sloping strip Σ of width $\rho\pi$. Since the boundary of B approaches one or the other edge of Σ as the preimage z approaches the point 1 from above or below, and since (for $\cos\phi \neq 0$) the strip Σ meets vertical lines in segments of length $\rho\pi|\sec\phi|$, the function (1) is univalent if and only if $\rho\pi|\sec\phi| \leq 2\pi$, that is, if and only if $\rho \leq |2\cos\phi|$. This proves the lemma. We note that (1) maps D onto a Jordan domain if and only if $\cos\phi > 0$.

Returning to the theorem, we note that

$$f'(z) = A_1 \exp[(\mu - 1)\log(1 - z)],$$

$$[f'(z)]^{\alpha} = A_2 \exp[\alpha(\mu - 1)\log(1 - z)],$$

$$f_{\alpha}(z) = A_3 \exp\{[\alpha(\mu - 1) + 1]\log(1 - z)\} + A_4,$$

where the A_k are constants and are therefore irrelevant to the question of univalence. By the lemma, f_α is univalent if and only if the point $w = \alpha(\mu - 1) + 1$ lies in one of the closed disks $|w + 1| \leq 1$ and $|w - 1| \leq 1$; that is, the function (1) serves the purpose of Theorem 2 if and only if μ satisfies one of the conditions (2) while w lies in the intersection W of the sets $|w + 1| > 1$ and $|w - 1| > 1$.

Since $\mu = 1 + (w - 1)/\alpha$, the conditions (2) are equivalent to

$$|w - 1 + 2\alpha| \leq |\alpha|, \qquad |w - 1| \leq |\alpha|, \tag{3}$$

respectively. Therefore the theorem is proved if we can show that whenever $|\alpha| > 1/3$ and $\alpha \neq 1$, one of the closed disks (3) meets the domain W. Since the center of the first disk in (3) lies at a distance 2α from the point $w = 1$, it is geometrically obvious that the disk meets the domain W if and only if $|\alpha| > 1/3$ and $\alpha \neq 1$. (The second disk, meeting W if and only if $|\alpha| > 1$, does not extend the effective range of our example.) The proof of Theorem 2 is complete.

While we have no example of a function f in S such that, whenever $|\alpha| > 1/3$ and $\alpha \neq 1$, the function f_α is not univalent, one of our functions covers a fairly large portion of the range. If in (1) we take $\mu = -2$, then f_α is univalent only when the point $1 - 3\alpha$ lies outside of W, that is, whenever

$$|1 - 3\alpha + 1| \leq 1 \qquad \text{or} \qquad |1 - 3\alpha - 1| \leq 1;$$

in other words, f_α is not univalent when α lies outside of the two closed disks with respective centers 0 and 2/3 and with radius 1/3.

It seems highly plausible that if $f(z) = \exp g(z)$, where g is a univalent function that maps D onto a slowly oscillating strip Σ of vertical height 2π, such as the strip bounded by the segment $[1 - \pi i, 1 + \pi i]$ and the two curves

$$y = \pm\pi + x(\log x)\sin(\log x) \qquad (x > 1),$$

then f_α is univalent if and only if $|\alpha| \leq 1/3$ or $\alpha = 1$. The principal problem that remains is to determine whether f_α is univalent if $f \in S$ and $(\sqrt{5} - 2)/3 < \alpha \leq 1/3$.

REFERENCES

1. P. L. Duren, H. S. Shapiro, and A. L. Shields, *Singular measures and domains not of Smirnov type*, Duke Math. J. (to appear).

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