DOUBLY STOCHASTIC MEASURES AND MARKOV OPERATORS

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1. INTRODUCTION

Let $(X, \mathscr{F}, m)$ be a totally σ-finite measure space, and let $(X^2, \mathscr{F}^2)$ denote the direct product of $(X, \mathscr{F})$ with itself. A measure λ on $(X^2, \mathscr{F}^2)$ is said to be *doubly stochastic* if

$$\lambda(A \times X) = \lambda(X \times A) = m(A) \tag{1}$$

for all $A \in \mathscr{F}$. For the case where X is the real line and m is Lebesgue measure, J. E. L. Peck has shown [6] that every doubly stochastic measure is a limit of convex combinations of permutation measures in a certain topology defined on the set M of all such measures. A *permutation measure* can be characterized as a measure on $(X^2, \mathscr{F}^2)$ that is concentrated on the graph of an invertible measure-preserving transformation ϕ of $(X, \mathscr{F})$, in other words, as a measure λ satisfying a condition of the form

$$\lambda(A \times B) = m(A \cap \phi^{-1} B). \tag{2}$$

In the case of a finite, homogeneous measure space $(X, \mathscr{F}, m)$, the present author has shown that every doubly stochastic measure is the limit of permutation measures, [1, Theorem 1], and that the relation

$$\lambda(A \times B) = (\chi_A, T\chi_B) \qquad (A, B \in \mathscr{F}) \tag{3}$$

establishes a one-to-one correspondence between the set M of doubly stochastic measures λ and the set of Markov operators T. Here we denote as usual the characteristic function of the set A by χ_A, and we use the symbol (f, g) for

$$\int_X f(x)\, g(x)\, m(dx),$$

the inner product in the real Hilbert space $L_2(X)$. A *Markov operator* T is a positive linear operator on $L_2(X)$ such that $T1 = T^* 1 = 1$. The topology on M is the weak operator topology for operators on $L_2(X)$, and M is compact in this topology (see [1]).

In the case of an infinite measure space, not every Markov operator (as defined below) is related to a doubly stochastic measure by (3), and we are led to the consideration of doubly substochastic measures. Moreover, M is not compact in the weak operator topology. Nevertheless, we shall show (Theorems 4 and 6) that every doubly stochastic measure is a limit of permutation measures, thus improving on the above-mentioned result of Peck. We shall also investigate the relationship between Markov operators and doubly stochastic measures.

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2. DEFINITIONS

We shall restrict our attention to the real half-line $X = [0, \infty)$ and to Lebesgue measure m on the class $\mathscr{F}$ of Borel subsets of X. The extension of our results to any infinite, σ-finite, homogeneous, nonatomic measure space will be evident, because any such space is isomorphic to a countable union of products of the unit interval with the product Lebesgue measure (see [5]). Thus it may be assumed that X is σ-compact, that $\mathscr{F}$ is the class of Borel sets in X, and that m is a regular Borel measure. Moreover, any two subsets of X with the same measure are related by an invertible measure-preserving transformation. In particular, our results are valid for n-dimensional Euclidean space with n-dimensional Lebesgue measure. Nonhomogeneous spaces are excluded from our consideration for the same reasons as in [1].

Definition 1. Let Λ denote the set of positive measures λ on $(X^2, \mathscr{F}^2)$ such that

$$\lambda(A \times B) \leq \min\{m(A), m(B)\} \tag{4}$$

for all $A, B \in \mathscr{F}$. Let M denote the set of all $\lambda \in \Lambda$ such that (1) holds for all $A \in \mathscr{F}$. Elements of M are called *doubly stochastic*, and elements of Λ, *doubly substochastic*.

We shall show (Theorem 3) that Λ can be identified with a set of operators on $L_2(X)$. The topologies of interest on Λ will be the weak operator (WO) and strong operator (SO) topologies.

Definition 2. Let Λ_1 denote the set of positive linear operators T on $L_\infty(X)$ such that

$$\|Tf\|_\infty \leq \|f\|_\infty, \qquad \|Tg\|_1 \leq \|g\|_1 \tag{5}$$

for all $f \in L_\infty(X)$ and all $g \in L_1(X) \cap L_\infty(X)$. Let M_1 denote the set of all $T \in \Lambda_1$ such that

$$T1 = 1, \qquad \int_X Tg\,dm = \int_X g\,dm \tag{6}$$

for all $g \in L_1(X) \cap L_\infty(X)$. The elements of M_1 are called *Markov operators* (with invariant measure m), and the elements of Λ_1, *sub-Markov operators* (with subinvariant measure m).

The topology of interest on Λ_1 is the weakest topology for which all of the linear functionals (f, Tg) and (g, Tf) are continuous in T for fixed $f \in L_\infty(X)$ and $g \in L_1(X) \cap L_\infty(X)$. We shall call this topology the *Peck* (P) *topology.*

Note that condition (5) and the Riesz convexity theorem imply that for each p $(1 \leq p < \infty)$, T can be extended in a unique way to a bounded linear operator on $L_p(X)$, with $\|Tf\|_p \leq \|f\|_p$ for all $f \in L_p(X)$.

The consideration of doubly substochastic measures is suggested by D. G. Kendall's paper [4] on doubly stochastic matrices. The topology used by Kendall and that used by B. A. Rattray and J. E. L. Peck [7] are closely related to the P-topology. The topology introduced by Peck in [6] is weaker than the P-topology and stronger than the WO-topology. However, the possibility that all three coincide on M is not ruled out.

3. STATEMENT OF RESULTS

The fundamental lemma of [1] for finite measure spaces takes the following form, in the present case.

LEMMA. *Let* $\mathscr{F}_0^2$ *denote the ring of finite unions of rectangles* $A \times B$, *where* A *and* B *are bounded Borel subsets of* X. *Let* λ *be a finitely additive, nonnegative set function on* $\mathscr{F}_0^2$ *satisfying* (4) *for all bounded Borel sets* A *and* B. *Then* λ *is countably additive and regular, hence has a unique extension to a doubly substochastic measure on* $\mathscr{F}^2$.

Proof. The proof of regularity and hence of countable additivity proceeds exactly as in [1]. The existence and uniqueness of the extension follow from the Hahn extension theorem [3, p. 54]. The substochastic nature of λ when A or B is unbounded follows by the regularity of $\lambda(A \times B)$ in A or B individually with the other held fixed.

Note that the proof of regularity of λ breaks down if rectangles $A \times B$ are considered where A or B is allowed to have infinite measure. Thus a finitely additive, nonnegative set function λ_0, defined on all measurable rectangles $A \times B$ ($A, B \in \mathscr{F}$) and satisfying (1) for all $A \in \mathscr{F}$, determines, according to the above lemma, a doubly substochastic measure λ on $\mathscr{F}^2$ that coincides with λ_0 on $\mathscr{F}_0^2$. In general, however, λ need not be doubly stochastic.

On the other hand, if ϕ is a measure-preserving transformation of $(X, \mathscr{F}, m)$, and if λ_ϕ is the corresponding set function defined on measurable rectangles by (2), then λ_ϕ is doubly stochastic. For if $\{B_n\}$ is a sequence of bounded Borel sets with $B_n \uparrow X$, then

$$\lambda_\phi(A \times B_n) = m(A \cap \phi^{-1} B_n) \rightarrow m(A)$$

for each $A \in \mathscr{F}$. Thus $\lambda_\phi(A \times X) = m(A)$. Similarly, $\lambda_\phi(X \times A) = m(A)$.

Let us denote by Φ the set of invertible measure-preserving transformations ϕ of $(X, \mathscr{F}, m)$ onto itself. As usual, we identify transformations that coincide almost everywhere. If $\phi_1 \neq \phi_2$, then there must exist a set B of finite measure such that $m(\phi_1^{-1} B \bigtriangleup \phi_2^{-1} B) > 0$, where $\bigtriangleup$ denotes the symmetric difference. We may assume, without loss of generality, that $m(\phi_1^{-1} B) > m(\phi_1^{-1} B \cap \phi_2^{-1} B)$. Let $A = \phi_1^{-1} B$. Then

$$\lambda_{\phi_1}(A \times B) = m(\phi_1^{-1} B) > m(\phi_1^{-1} B \cap \phi_2^{-1} B) = \lambda_{\phi_2}(A \times B),$$

so that $\lambda_{\phi_1} \neq \lambda_{\phi_2}$. Thus Φ may be identified with the set of all λ_ϕ ($\phi \in \Phi$). From the preceding paragraph it follows that $\Phi \subset M \subset \Lambda$. We can now state the main results as a sequence of six theorems.

THEOREM 1. *Each* $T \in \Lambda_1$ *determines a unique* $\lambda \in \Lambda$ *such that*

$$\lambda(A \times B) = (\chi_A, T\chi_B)$$

for all bounded Borel sets A *and* B. *The correspondence is many-to-one.*

THEOREM 2. *Each* $\lambda \in \Lambda$ *determines a unique* $T \in \Lambda_1$ *such that*

$$\lambda(A \times B) = (\chi_A, T\chi_B)$$

for all Borel sets A *and* B *such that at least one of them is bounded. The correspondence is one-to-one. If* $\lambda \in M$, *then* $T \in M_1$.

Note that, according to Theorem 2, the correspondence of Theorem 1 exhausts Λ.

THEOREM 3. *The correspondence of Theorem* 1 *between* $\lambda \in \Lambda$ *and* $T \in \Lambda_1$ *induces a one-to-one correspondence between* λ *and the restriction of* T *to an operator on* $L_2(X)$. Λ *is compact in the* WO-*topology.*

THEOREM 4. Φ *is dense in* Λ *in the* WO-*topology.* Λ *is the closed convex hull of* Φ *in the* SO-*topology.* M *is not closed in either topology.*

According to Theorem 2, we may consider $\Lambda \subset \Lambda_1$ and $M \subset M_1$. Recall that the P-topology is defined on Λ_1 and that $M_1 \subset \Lambda_1$.

THEOREM 5. Λ_1 *and* M_1 *are compact in the* P-*topology.* M *is not closed in this topology.*

THEOREM 6. Φ *is dense in* M *in the* P-*topology.*

4. PROOFS OF THEOREMS 1 TO 6

Proof of Theorem 1. Since the right side of (3) is additive for disjoint rectangles, λ can be uniquely extended by additivity to a nonnegative, finitely additive set function on $\mathscr{F}_0^2$. Since $\|T\chi_B\|_1 \leq m(B)$ and $\|T\chi_B\|_\infty \leq 1$, we have the relations

$$\lambda(A \times B) \leq \|\chi_A\|_1 \|T\chi_B\|_\infty \leq m(A)$$

and

$$\lambda(A \times B) \leq \|\chi_A\|_\infty \|T\chi_B\|_1 \leq m(B)$$

for all bounded Borel sets A and B. According to the lemma, λ has a unique extension to an element of Λ.

For each $f \in L_\infty(X)$, let

$$Tf(x) = \operatorname*{LIM}_{y \to \infty} f(y),$$

where LIM denotes a Banach limit [2, p. 73]. Then T is a positive linear operator on $L_\infty(X)$ with $T1 = 1$ and $Tf = 0$ for all $f \in L_1(X)$. Thus $T \in \Lambda_1$. The corresponding measure λ, determined by (3) for bounded Borel sets, is identically 0. Thus the correspondence is many-to-one, as asserted.

Proof of Theorem 2. Suppose that $\lambda \in \Lambda$ and $g \in L_\infty(X)$. For each simple function f on $(X, \mathscr{F}, m)$ with compact support, set

$$G(f) = \int_{X^2} f(x)g(y)\lambda(dx, dy). \tag{7}$$

Since $\lambda(A \times X) \leq m(A)$ for $A \in \mathscr{F}$, it follows that

$$|G(f)| \leq \|g\|_\infty \int_{X^2} |f(x)|\lambda(dx, dy) \leq \|g\|_\infty \int_X |f(x)|\, m(dx) = \|g\|_\infty \|f\|_1. \tag{8}$$

Thus (7) defines a bounded linear functional G on a dense subset of $L_1(X)$. It follows that there exists a function $Tg \in L_\infty(X)$ such that

$$(f, Tg) = \int_{X^2} f(x)\, g(y)\, \lambda(dx, dy) \tag{9}$$

for all $f \in L_1(X)$. If $g \geq 0$, then G is a positive functional, so that $Tg \geq 0$. It follows that T is a positive linear operator on $L_\infty(X)$. From the inequality (8) it follows that $\|Tg\|_\infty = \|G\| \leq \|g\|_\infty$. Similarly, if g is a simple function with compact support and $f \in L_1(X) \cap L_\infty(X)$, then equation (9) gives the relations

$$|(f, Tg)| \leq \|f\|_\infty \int_{X^2} |g(y)|\, \lambda(dx, dy) \leq \|f\|_\infty \|g\|_1, \tag{10}$$

since $\lambda(X \times A) \leq m(A)$ for all $A \in \mathscr{F}$. It follows that (10) holds for all

$$f, g \in L_1(X) \cap L_\infty(X).$$

In particular,

$$\int_A |Tg(x)|\, m(dx) \leq \|g\|_1$$

for all bounded Borel sets A. Hence $\|Tg\|_1 \leq \|g\|_1$, and T is a sub-Markov operator.

From (9), we see that

$$(\chi_A, T\chi_B) = \lambda(A \times B)$$

whenever $A, B \in \mathscr{F}$ and A is bounded. If B is bounded, then $T\chi_B \in L_1(X)$, so that each side of the above equation is countably additive in A. Hence equality holds for all $A \in \mathscr{F}$. The biuniqueness of the correspondence now follows from Theorem 1 and equation (9), since T is uniquely determined by the values of (f, Tg) for $f \in L_1(X)$ and $g \in L_\infty(X)$.

If in particular $\lambda \in M$, then

$$(\chi_A, T1) = \lambda(A \times X) = m(A)$$

for all bounded Borel sets A. It follows that $(f, T1) = (f, 1)$ for all $f \in L_1(X)$, and so $T1 = 1$. Likewise,

$$\int_X Tf\, dm = (1, Tf) = (1, f) = \int_X f\, dm$$

for all $f \in L_1(X) \cap L_\infty(X)$. Thus $T \in M_1$.

Proof of Theorem 3. Suppose that $T_1, T_2 \in \Lambda_1$ have the same restriction to $L_2(X) \cap L_\infty(X)$. Since characteristic functions of bounded Borel sets belong to $L_2(X)$, it follows that T_1 and T_2 determine the same measure λ under the correspondence of Theorem 1. Conversely, if T_1 and T_2 determine the same measure, then they have the same restriction, since T, considered as an operator on $L_2(X)$, is determined by the values $(\chi_A, T\chi_B)$, where A and B range over all bounded Borel sets.

Thus Λ is identified with a subset of the unit sphere in the space of all bounded linear operators on $L_2(X)$, namely, the set of positive operators T such that

$$(\chi_A, T\chi_B) \leq \min \{m(A), m(B)\}$$

for all bounded Borel sets A and B. It follows that Λ is closed and hence compact in the WO-topology [2, p. 512].

Proof of Theorem 4. A basis for the weak operator topology is given by all sets of the form

$$\{T: |(f_k, Tg_k) - (f_k, T_0 g_k)| < \varepsilon, \ k = 1, \cdots, n\}, \tag{11}$$

where f_k and g_k run through a dense subset of $L_2(X)$, T_0 is the operator corresponding to some $\lambda_0 \in \Lambda$, and $\varepsilon > 0$. In particular, we may assume that f_k and g_k are continuous with compact support. In this case they are bounded, and by the arbitrariness of ε in (11) we may assume that they are bounded by 1. We shall show that there always exists a measure-preserving transformation ϕ such that the operator corresponding to λ_ϕ belongs to the set (11). According to (9), this means that we must show that

$$\left| \int_{X^2} h_k d\lambda_\phi - \int_{X^2} h_k d\lambda_0 \right| < \varepsilon \qquad (k = 1, \cdots, n), \tag{12}$$

where $h_k(x, y) = f_k(x) g_k(y)$ $(k = 1, \cdots, n)$.

Let K be a compact interval such that each of the f_k and each of the g_k vanishes outside K. Then the h_k vanish outside $K \times K$ and, by uniform continuity, there exist disjoint intervals $X_1, \cdots, X_s$ such that $K = \bigcup_{i=1}^s X_i$ and the oscillation of each h_k is less than $\varepsilon/4m(K)$ on each rectangle $X_i \times X_j$ $(i, j = 1, \cdots, s)$.

Now for $i = 1, \cdots, s$ we have the relations

$$\sum_{j=1}^{s} \lambda_0(X_i \times X_j) = \lambda_0(X_i \times K) \leq m(X_i).$$

Thus there exist disjoint intervals X_{ij} $(j = 1, \cdots, s + 1)$ such that

$$X_i = \bigcup_{j=1}^{s+1} X_{ij} \quad \text{and} \quad m(X_{ij}) = \lambda_0(X_i \times X_j) \quad (j = 1, \cdots, s).$$

Let X_{s+1} be an interval disjoint from K, with $m(X_{s+1}) = m(K) - \lambda_0(K \times K)$, and choose disjoint subintervals $X_{s+1,j}$ such that

$$X_{s+1} = \bigcup_{j=1}^{s} X_{s+1,j} \quad \text{and} \quad m(X_{s+1,j}) = m(X_j) - \lambda_0(K \times X_j) \quad (j = 1, \cdots, s).$$

Likewise, for $j = 1, \cdots, s$ there exist disjoint intervals Y_{ij} such that

$$X_j = \bigcup_{i=1}^{s+1} Y_{ij} \quad \text{and} \quad m(Y_{ij}) = \lambda_0(X_i \times X_j) \quad (i = 1, \cdots, s),$$

and there exist disjoint intervals $Y_{i,s+1}$ such that

$$X_{s+1} = \bigcup_{i=1}^{s} Y_{i,s+1} \quad \text{and} \quad m(Y_{i,s+1}) = m(X_i) - \lambda_0(X_i \times K) \quad (i = 1, \cdots, s).$$

Clearly, $m(X_{ij}) = m(Y_{ij})$ for $i, j = 1, \cdots, s$. Moreover, for $i = 1, \cdots, s$,

$$m(X_{i,s+1}) = m(X_i) - \sum_{j=1}^{s} m(X_{ij}) = m(X_i) - \lambda_0(X_i \times K) = m(Y_{i,s+1}).$$

Likewise, $m(X_{s+1,j}) = m(Y_{s+1,j})$ $(j = 1, \cdots, s)$. Let ϕ be an invertible measure-preserving transformation of $(X, \mathscr{F}, m)$ that maps X_{ij} onto Y_{ij} $(i, j = 1, \cdots, s+1;\ i + j < 2s + 2)$ and maps the complement of $K \cup X_{s+1}$ onto itself. We shall show that the measure λ_ϕ defined by (2) assigns the same measure to each rectangle $X_i \times X_j$ as does λ_0.

Indeed, ϕ maps $X_i = \bigcup_j X_{ij}$ onto $\bigcup_j Y_{ij}$, and $Y_{ij} \subset X_j$, so that

$$\lambda_\phi(X_i \times X_j) = m(X_i \cap \phi^{-1} X_j) = m(X_{ij}) = \lambda_0(X_i \times X_j).$$

It follows that

$$\left| \int_{X^2} h_k \, d\lambda_\phi - \int_{X^2} h_k \, d\lambda_0 \right| \le \sum_{i,j=1}^{s} \left| \int_{X_i \times X_j} h_k \, d\lambda_\phi - \int_{X_i \times X_j} h_k \, d\lambda_0 \right|$$
$$\le 2(\varepsilon/4m(K))\lambda_0(K \times K) < \varepsilon.$$

We have proved that Φ is dense in Λ.

Since Λ is clearly convex and since convex sets have the same closure in the WO- and the SO-topologies [2, p. 477], it follows that Λ is the closed convex hull of Φ in the SO-topology. Finally, since $\Phi \subset M \subset \Lambda$, it follows that M is not closed in either topology.

Proof of Theorem 5. The defining conditions for Λ_1 may be written

$$|(f, Tg)| \le \|f\|_1 \|g\|_\infty \quad \text{and} \quad |(g, Tf)| \le \|f\|_1 \|g\|_\infty$$

for all $f \in L_1(X) \cap L_\infty(X)$, $g \in L_\infty(X)$. The defining conditions for M_1 are

$$(f, T1) = (f, 1) \quad \text{and} \quad (1, Tf) = (1, f)$$

for all $f \in L_1(X) \cap L_\infty(X)$. Since each of these properties as well as the bilinearity and positivity of the expression (f, Tg) in f and g are obviously preserved under a passage to the limit in the P-topology, it follows by the Tychonoff product theorem that both Λ_1 and M_1 are compact.

The WO-topology on Λ is determined by all functionals of the form (f, Tg) for $f, g \in L_2(X)$. The topology determined by those functionals for which f and g are characteristic functions of bounded Borel sets is weaker; but it is nevertheless a Hausdorff topology, since each $\lambda \in \Lambda$ is uniquely determined by its values on $\mathscr{F}_0^2$. Since Λ is compact in the WO-topology, the weaker topology must coincide with the WO-topology on Λ. In particular, the two topologies coincide on M. Since characteristic functions of bounded Borel sets belong to $L_1(X) \cap L_\infty(X)$, it follows that the WO-topology is weaker than the P-topology on M. (The possibility that they coincide is not excluded.) Now, if M were closed in the P-topology as a subset of M_1, then it would be compact in this topology and hence also in the weaker WO-topology. But M is not WO-closed in Λ. Hence M is not P-closed.

Proof of Theorem 6. This proof is similar to the proof of Theorem 4, except that we cannot restrict our attention to functions vanishing outside of some fixed compact set.

In place of the set (11) we consider a general P-basic set of the form

$$(13) \quad \{T: |(f_k, Tg_k) - (f_k, T_0 g_k)| < \varepsilon, \ |(g_k, Tf_k) - (g_k, T_0 f_k)| < \varepsilon, \ k = 1, \cdots, \ell\},$$

where $f_1, \cdots, f_\ell$ are continuous, bounded by 1, and vanish outside of the compact interval K, and where $g_1, \cdots, g_\ell$ are measurable functions bounded by 1. According to Lusin's theorem [3, p. 242], there exist compact sets $K_1, K_2, \cdots$ such that $K_n \subset [n - 1, n]$, each g_k is continuous (hence uniformly continuous) on K_n, and $m([n - 1, n] - K_n) < \varepsilon/2^{n+2}$. It follows that each interval $[n - 1, n]$ is the union of disjoint subintervals J_{np} such that the oscillation of each g_k is less than $\sqrt{(\varepsilon/4m(K))}$ on each set $J_{np} \cap K_n$.

Let N be the union of the sets $[n - 1, n] - K_n$ $(n = 1, 2, \cdots)$. Then

$$(13) \quad m(N) \leq \sum_{n=1}^{\infty} \varepsilon/2^{n+2} = \varepsilon/4,$$

and X is the union of a countable sequence $\{X_i\}$ of disjoint intervals such that

$$(14) \quad |g_k(x) - g_k(y)| < \sqrt{(\varepsilon/4m(K))}$$

for all $x, y \in X_i - N$ $(k = 1, \cdots, \ell;\ i = 1, 2, \cdots)$. By the uniform continuity of the f_k, we may assume, in addition, that

$$(15) \quad |f_k(x) - f_k(y)| < \sqrt{(\varepsilon/4m(K))}$$

for all $x, y \in X_i$ $(k = 1, \cdots, \ell;\ i = 1, 2, \cdots)$. Now let

$$h_k(x, y) = f_k(x) g_k(y) \quad \text{and} \quad h_k^*(x, y) = g_k(x) f_k(y).$$

Then, from (14), (15), and the fact that each of f_k and g_k is bounded by 1, it follows that the oscillation of h_k and the oscillation of h_k^* are both bounded by $\varepsilon/4m(K)$ on the sets $X_i \times (X_j - N)$ and $(X_i - N) \times X_j$, respectively $(i, j = 1, 2, \cdots)$. Moreover, each h_k vanishes outside of $K \times X$, and each h_k^* vanishes outside of $X \times K$.

Now suppose that $T_0 \in M$, and let λ_0 be the corresponding doubly stochastic measure. Then there exist disjoint intervals X_{ij} and disjoint intervals Y_{ij} such that

$$X_i = \bigcup_{j=1}^{\infty} X_{ij}, \quad X_j = \bigcup_{i=1}^{\infty} Y_{ij},$$

and

$$m(X_{ij}) = m(Y_{ij}) = \lambda_0(X_i \times X_j)$$

for all i, j = 1, 2, $\cdots$. Let ϕ be an invertible measure-preserving transformation of X onto itself such that $\phi(X_{ij}) = Y_{ij}$ for all i and j. We shall show that the operator T_ϕ corresponding to the measure λ_ϕ belongs to the set (13).

Indeed, it follows as in the proof of Theorem 4 that $\lambda_\phi(X_i \times X_j) = \lambda_0(X_i \times X_j)$ for each i and j. Recalling that h_k is bounded by 1, we see that

$$\left| \int_{X_i \times X_j} h_k \, d\lambda_\phi - \int_{X_i \times X_j} h_k \, d\lambda_0 \right| \leq 2(\varepsilon/4m(K))\lambda_0((X_i \cap K) \times (X_j - N)) + 2\lambda_0((X_i \cap K) \times (X_j \cap N)).$$

Thus

$$\text{(16)} \quad \left| \int_{X^2} h_k \, d\lambda_\phi - \int_{X^2} h_k \, d\lambda_0 \right| \leq (\varepsilon/2m(K))\lambda_0(K \times X) + 2\lambda_0(K \times N) \leq (\varepsilon/2m(K))m(K) + 2m(N) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Similarly,

$$\text{(17)} \quad \left| \int_{X^2} h_k^* \, d\lambda_\phi - \int_{X^2} h_k^* \, d\lambda_0 \right| < \varepsilon .$$

The inequalities (16) and (17) for $k = 1, \cdots, \ell$ are clearly equivalent to the statement that T_ϕ belongs to the set (13). This completes the proof of Theorem 6.

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