BOREL SETS OF STATES AND OF REPRESENTATIONS

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Let $\mathscr{A}$ be a separable c*-algebra and Σ its set of states. In [6], S. Sakai shows that the set of states of $\mathscr{A}$ giving rise to primary representations is an analytic set, and he asks whether this set is in fact a Borel set. We show that this is indeed the case, and we answer similar questions for other naturally defined classes of states as well as for the corresponding classes of representations.

Let H_n be a Hilbert space of dimension n, and let $\mathfrak{R}_n$ be the set of nowhere trivial representations of $\mathscr{A}$ in H_n, topologized by the weakest topology for which the mapping $\rho \to \rho_A x$ is continuous for each $A \in \mathscr{A}$ and $x \in H_n$. It is known [1] that $\mathfrak{R}_n$ is a Polish space.

Also, let

$$\mathfrak{R} = \bigcup_{0 \leq n \leq \aleph_0} \mathfrak{R}_n,$$

and let $\mathfrak{R}$ be topologized by making each $\mathfrak{R}_n$ open in $\mathfrak{R}$. Now let

$$\mathfrak{Z}_n = \{(\rho, x): \rho \in \mathfrak{R}_n, \; x \in H_n, \; x \text{ cyclic for } \rho\}.$$

$\mathfrak{Z}_n$ is a subset of $\mathfrak{R}_n \times H_n$.

LEMMA 1. *$\mathfrak{Z}_n$ is of type G_δ.*

Proof. If $\{A_k\}$ is dense in $\mathscr{A}$ and $\{x_j\}$ in H_n, then

$$\mathfrak{Z}_n = \bigcap_{i,j} \bigcup_k \left\{(\rho, x): \|\rho_{A_k} x - x_j\| < \frac{1}{i}\right\}.$$

But for fixed k, the map $(\rho, x) \to \rho_{A_k} x$ is continuous, since

$$\|\rho_{A_k} x - \rho'_{A_k} x'\| \leq \|\rho_{A_k}(x - x')\| + \|\rho_{A_k} x - \rho'_{A_k} x\|$$
$$\leq \|A_k\| \, \|x - x'\| + \|\rho_{A_k} x - \rho_{A_k} x'\|.$$

Thus $(\rho, x) \to \|\rho_{A_k} x - x_j\|$ is continuous. Q.E.D.

Now set $\mathfrak{Z} = \bigcup_n \mathfrak{Z}_n$, and topologize $\mathfrak{Z}$ by making each $\mathfrak{Z}_n$ open. Then $\mathfrak{Z}$ is again Polish. Define a map $s: \mathfrak{Z} \to \Sigma$ by $s(\rho, x)(A) = (\rho_A x, x)$.

LEMMA 2. *The map s is continuous and onto.*

Proof. That s is onto is clear from the well-known relation between states and cyclic representations [8]. As for continuity, we observe that

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$$|(\rho_A x, x) - (\rho'_A x', x')| \leq |(\rho_A - \rho'_A)x, x)| + |(\rho'_A(x - x'), x)| + |(\rho'_A x', x - x')|$$
$$\leq (\|\rho_A x - \rho'_A x\| + 2\|A\|\,\|x - x'\|)\,\|x\|. \qquad \text{Q.E.D.}$$

From this, it is possible to show that various sets in Σ are Borel sets. Namely, if Λ is a set in Σ, and $s^{-1}(\Lambda)$ is analytic, then Λ is the continuous image of an analytic set, hence is itself analytic. If $s^{-1}(\Lambda)$ is actually a Borel set, then $\mathcal{B} - s^{-1}(\Lambda)$ is also analytic, so that $\Sigma - \Lambda = s(\mathcal{B} - s^{-1}(\Lambda))$ is also analytic, and hence Λ is a Borel set.

Hereafter, for $\rho \in \mathfrak{R}(H_n)$, we denote by $\mathcal{M}_\rho$ the strong closure of the algebra $\rho(\mathcal{A})$ on H_n.

LEMMA 3. *The set of pairs* (ρ, B) *in* $\mathfrak{R}_n \times \mathcal{B}(H_n)$ *with* $B \in \mathcal{M}_\rho$ *is a Borel set.*

Proof. Let $A_1, A_2, \cdots$ and $x_1, x_2, \cdots$ be dense in $\mathcal{A}$ and in the unit ball of H_n, respectively. Now,

$$\|\rho(A_j)\| \leq 1 \Leftrightarrow \|\rho(A_j)x_k\| \leq 1 \text{ for all } k.$$

Using a theorem of Kaplansky in [4], we see that $\{\rho(A_j): \|\rho(A_j)\| \leq 1\}$ is dense in the unit ball of $\mathcal{M}_\rho$. Thus

$$[B \text{ is in the unit ball of } \mathcal{M}_\rho]$$
$$\Leftrightarrow \Big[\text{for each } n\ \exists\, m \text{ such that } \|\rho(A_m)\| \leq 1$$
$$\text{and } \Big(j \leq n \Rightarrow \|(\rho(A_m) - B)x_j\| < \frac{1}{n}\Big)\Big].$$

Therefore the set of pairs (ρ, B) in question is

$$\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \Big\{(\rho, B): \|\rho(A_m)\| \leq 1 \text{ and } \|(\rho(A_m) - kB)x_j\| < \frac{1}{n}\Big\}.$$

LEMMA 4. *The set* $\mathfrak{R}^P \cap \mathfrak{R}^{II}$ *of primary representations of type* II *is an analytic set.*

Proof. We see this by copying the argument of J. Schwartz in [7]. In the one place where this cannot be done, namely his condition (g) on p. 250, we substitute our Lemma 3.

LEMMA 5. *The set* $\mathfrak{R}^P \cap \mathfrak{R}^f$ *of primary representations of finite type is a Borel set.*

Proof. We show that both $\mathfrak{R}^P \cap \mathfrak{R}^f$ and $\mathfrak{R} - \mathfrak{R}^f$ are analytic. Since it was shown by Dixmier [2] that $\mathfrak{R}^P$ is a Borel set, it will then follow that $\mathfrak{R}^P \cap \mathfrak{R}^f$ is a Borel set.

Now, in general, the trace in a factor of finite type may be written as a finite linear combination of vector states (see [5, Theorem III]). Thus, if $\rho \in \mathfrak{R}_n \cap \mathfrak{R}^P$, then $\mathcal{M}_\rho$ is finite if and only if $\exists\, x_1, \cdots, x_k \in H_n$ with

$$\sum_{i=1}^{k} \|x_i\|^2 = 1 \quad \text{and} \quad \sum_{i=1}^{k} (\rho(A_\ell)\rho(A_m)x_i, x_i) = \sum_{i=1}^{k} (\rho(A_m)\rho(A_\ell)x_i, x_i)$$

($A_1, A_2, \cdots$ being a preassigned uniformly dense sequence in $\mathcal{A}$). Then $\mathfrak{R}_n \cap \mathfrak{R}^P \cap \mathfrak{R}^f$ is the union over $k = 1, 2, \cdots$ of the projection in $\mathfrak{R}_n$ of the following subset of $(\mathfrak{R}_n \cap \mathfrak{R}^P) \times \underbrace{H_n \times \cdots \times H_n}_{k \text{ times}}$:

$$\left\{(\rho, x_1, \cdots, x_k): \sum_{i=1}^{k} \|x_i\|^2 = 1, \right.$$
$$\left. \sum_{i=1}^{k} (\rho(A_\ell A_m)x_i, x_i) = \sum_{i=1}^{k} (\rho(A_m A_\ell)x_i, x_i) \text{ for all } \ell, m \right\},$$

which is clearly a Borel set.

As for $\mathfrak{R} - \mathfrak{R}^f$: this is defined by the presence of an infinite projection in $\mathcal{M}_\rho$. If $\rho \in \mathfrak{R}_n$ (and here, of course, only $n = \aleph_0$ can really occur) and $x_1, x_2, \cdots$ is a dense sequence in H_n, then there exists a partial isometry U in $\mathcal{M}_\rho$ with

$$UU^* < U^*U \quad \text{and} \quad UU^* \neq U^*U.$$

Thus $\mathcal{M}_\rho$ is *not* of finite type if and only if it contains a U such that

(1) $(U^*U)^2 = U^*U$,

(2) $(UU^*)^2 = UU^*$ (actually this follows from (1), but we throw it in to save trouble).

(3) $(U^*Ux_i, x_i) \geq (UU^*x_i, x_i)$ for all i, and the inequality is strict for at least *some* i.

Now, applying Lemma 3 again, we see that as ρ ranges over $\mathfrak{R}_n$,

$$\{(\rho, U): U \in \mathcal{M}_\rho \text{ and satisfies (1), (2), (3)}\}$$

is a Borel set, so its projection in $\mathfrak{R}_n$ is analytic and is precisely $\mathfrak{R}_n - \mathfrak{R}^f$.

Combining these facts with facts already known from [2] and [3], we summarize our results as follows.

THEOREM. *The primary representations of $\mathcal{A}$ and those of type I_n and II_1 form Borel sets in $\mathfrak{R}$, while those of type II_∞ form an analytic set and those of type III a co-analytic set. These statements hold also for the sets of states giving rise to the above classes of representations.*

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