ON CONSISTENCY OF ℓ-ℓ METHODS OF SUMMATION

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1. INTRODUCTION

Let ℓ and m denote the space of absolutely convergent series and the space of bounded sequences, respectively. Let A denote an infinite matrix defining a series-to-series transformation that preserves absolute convergence of series, and let ℓ_A denote its absolute summability field. We prove a number of results that relate with one another the concepts of perfectness, reversibility, and type M^*. We also prove some theorems giving conditions for absolute consistency of two matrix methods and for the existence of matrices A with the property that if $f \in \ell'_A$ (the dual space of ℓ_A), then there exists a matrix B such that $\ell_A \subseteq \ell_B$ and $B(x) = f(x)$ for $x \in \ell_A$. Finally, we extend to the class of perfect matrices a theorem that Macphail [2] proved for reversible matrices, and we demonstrate the existence of a nonreversible perfect matrix. Our results belong largely to a class of theorems due to Mazur [3], Mazur and Orlicz [4], Wilansky [6], and Zeller [9].

2. MATRIX MAPPINGS

Let $A = (a_{nk})$ and $x = \{x_k\}$ be a matrix and a sequence of complex numbers, respectively. We write formally

$$y_n \equiv A_n(x) \equiv \sum_k a_{nk} x_k, \tag{1}$$

and we say that the sequence x (and the corresponding series $\sum_k (x_k - x_{k-1})$ with $x_{-1} = 0$) is absolutely summable if each series in (1) converges and $\sum_n |y_n| < \infty$. We say the method is an ℓ-ℓ method provided $\sum_n |y_n| < \infty$ whenever $\sum_n |x_n| < \infty$, and that it is absolutely regular provided in addition $\sum_n y_n = \sum_n x_n$ whenever $\sum_n |x_n| < \infty$. Regarding these concepts the following theorem was proved by Knopp and Lorentz [1] and by Mears [3].

THEOREM (Knopp, Lorentz, Mears). *The matrix* A *defines an* ℓ-ℓ *method if and only if*

$$\sum_n |a_{nk}| \leq M \quad \text{(M independent of k)}. \tag{2}$$

The method A *is absolutely regular if and only if in addition to* (2) *it satisfies the condition*

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(3) $$\sum_n a_{nk} = 1 \quad (k = 0, 1, 2, \cdots).$$

Macphail concerned himself with reversible ℓ-ℓ methods, that is, with methods for which the equations $y_n = \sum_k a_{nk} x_k$ have exactly one solution $x \in \ell_A$ for each $y \in \ell$. This requirement essentially enables one to assume that the sequence space ℓ_A is a Banach space under an appropriate norm.

An application of Theorem 5 of [8, p. 230] shows that every $f \in \ell'_A$ may be evaluated as

(4) $$f(x) = \sum_n t_n A_n(x) + \sum_n a_n x_n$$

for some $t, a \in m$. Here the series $\sum_n a_n x_n$ converges for $x \in \ell_A$, and $A_n(x) = \sum_k a_{nk} x_k$. If we now set $A(x) = \sum_n \sum_k a_{nk} x_k$, then it is easily seen that $A \in \ell'_A$. If B is an ℓ-ℓ matrix such that $\ell_A \subseteq \ell_B$, then $B(x) = \sum_n \sum_k b_{nk} x_k$ is a continuous linear functional on ℓ_A, and hence

$$B(x) = \sum_n t_n A_n(x) + \sum_n a_n x_n$$

for some $t, a \in m$ and all $x \in \ell_A$. Following Macphail, we write $B \succ A$ if $\ell_A \subseteq \ell_B$, and $B \sim A$ if $B(x) = A(x)$ for $x \in \ell$. If $B(x) = A(x)$ for $x \in \ell_A \cap \ell_B$, we say B is *absolutely consistent* with A. The method A is said to be of *type* M^* if for every bounded sequence $\{g_n\}$ the condition

$$\sum_n g_n a_{nk} = 0 \qquad (k = 0, 1, 2, \cdots)$$

implies that $g_n = 0$ for $n = 0, 1, 2, \cdots$. The following theorem is due to Macphail [2].

THEOREM. *In order that a reversible ℓ-ℓ method* A *be absolutely consistent with every ℓ-ℓ method* B *such that* $B \succ A$ *and* $B \sim A$, *it is necessary and sufficient that* A *be of type* M^*.

We now define an ℓ-ℓ method to be *perfect* if ℓ is dense in ℓ_A in the seminorm topology [8, p. 226].

3. PRINCIPAL RESULTS

In this section we prove three theorems that make precise the results promised in the Introduction.

THEOREM 1. *Let* A *be an ℓ-ℓ matrix. In order that* A *be perfect, it is sufficient that*

(5) $$\sum_n t_n \sum_k a_{nk} x_k = \sum_k \sum_n t_n a_{nk} x_k$$

for each $t \in m$ *and* $x \in \ell_A$. *It is necessary that whenever* $t \in m$ *and the right-hand member of* (5) *exists for all* $x \in \ell_A$, *then* (5) *hold for each* $x \in \ell_A$.

Proof. Suppose (5) holds, and let f be an element of ℓ_A' that vanishes on ℓ. Then $f(e^p) = 0$, where $e^p = \{0, 0, \cdots, 1, 0, \cdots\}$ (the sequence of zeros except for a 1 in the pth coordinate). But since

$$f(x) = \sum_n t_n \sum_k a_{nk} x_k + \sum_k a_k x_k$$

for some t, a $\in$ m, we see that $0 = f(e^p) = \sum_n t_n a_{np} + a_p$ for all p. From this we deduce, since $\sum_p a_p x_p$ converges for $x \in \ell_A$, that

$$\sum_p a_p x_p = -\sum_p \left(\sum_n t_n a_{np}\right) x_p \qquad \text{for } x \in \ell_A,$$

and hence that

$$f(x) = \sum_n t_n \sum_k a_{nk} x_k - \sum_k \sum_n t_n a_{nk} x_k .$$

By (5), $f(x) = 0$ for $x \in \ell_A$. Hence ℓ is dense in ℓ_A.

For the proof of the second part of the theorem, we follow [7, p. 336]. Suppose A is perfect. Let $t \in m$ be such that the right-hand member of equation (5) converges for all $x \in \ell_A$. Then

$$f_t(x) = \sum_n t_n \sum_k a_{nk} x_k - \sum_k x_k \sum_n t_n a_{nk}$$

is a continuous linear functional on ℓ_A. Hence $f_t^\perp$ is closed. Therefore the set Q of points in ℓ_A such that equation (5) holds for each $t \in m$ for which the right-hand member of (5) exists for each $x \in \ell_A$ is closed. But clearly $\ell \subset Q$. Therefore $\bar{\ell} \subset Q$, and if A is perfect it follows that $Q = \ell_A$.

THEOREM 2. *A reversible ℓ-ℓ method is perfect if and only if it is of type* M^*.

Proof. Suppose A is of type M^*, and let $f \in \ell_A'$ and $f(x) = 0$ for all $x \in \ell$. Now for some t, a $\in$ m and $x \in \ell_A$,

$$f(x) = \sum_n t_n \sum_k a_{nk} x_k + \sum_k a_k x_k .$$

Since A is reversible, we can easily show that $a_k = 0$ for all k. Therefore

$$0 = f(e^p) = \sum_n t_n a_{np} \qquad \text{for all } p.$$

But since A is of type M^*, it follows that $t_n = 0$ for all n. Hence, $f(x) = 0$ for $x \in \ell_A$, and so A is perfect. Conversely, suppose A is perfect, and let $t \in m$ be such that $\sum_n t_n a_{nk} = 0$ for all k. If now A is not of type M^*, then $t_p \neq 0$ for at least one p. Choose $e^p \in \ell$, so that $\sum_n t_n e_n^p \neq 0$. By the reversibility of A, there exists a unique $x' \in \ell_A$ such that $e_n^p = \sum_k a_{nk} x_k'$. Therefore, we must have the relation

$$\sum_n t_n e_n^p = \sum_n t_n \sum_k a_{nk} x_k' \neq 0. \tag{6}$$

But $(t_n a_{nk})$ is an ℓ-ℓ method not weaker than A. Indeed,

$$\sum_n |t_n a_{nk}| \leq N \quad \text{(independent of k)},$$

and if $x \in \ell_A$, then, since $|t_n| \leq M$, say,

$$\sum_n |\sum_k t_n a_{nk} x_k| \leq M \sum_n |\sum_k a_{nk} x_k| < \infty.$$

Thus $\sum_n t_n \sum_k a_{nk} x_k$ defines an element of ℓ_A'. If $x \in \ell$ and $t \in m$, then $\sum_k \sum_n t_n a_{nk} x_k$ converges absolutely. Thus

$$\sum_n t_n \sum_k a_{nk} x_k = \sum_k \sum_n t_n a_{nk} x_k \quad \text{for } x \in \ell,$$

and since $\sum_n t_n a_{nk} = 0$ for all k, it follows that the functional $\sum_n t_n \sum_k a_{nk} x_k$ vanishes on ℓ. But since A is perfect, f(x) must vanish on ℓ_A, that is,

$$\sum_n \sum_k t_n a_{nk} x_k = 0$$

for $x \in \ell_A$. This contradicts (6), and hence A is of type M^*.

THEOREM 3. *A necessary and sufficient condition for an ℓ-ℓ method A to be absolutely consistent with every ℓ-ℓ method B for which $B \succ A$ and $B \sim A$ is that A be perfect.*

Proof. Suppose A is perfect and that $B \succ A$ and $B \sim A$. Then $F(x) = B(x) - A(x)$ is a continuous linear functional on ℓ_A that vanishes on ℓ. Hence $F(x) = 0$ for $x \in \ell_A$, that is, $B(x) = A(x)$ for $x \in \ell_A$. Conversely, suppose A is absolutely consistent with every B for which $B \succ A$ and $B \sim A$. In order to show that A is perfect, we use the following lemma.

LEMMA. *Let A be an ℓ-ℓ method and $f \in \ell_A'$. Then there exists a method B such that $B \succ A$ and $B(x) = f(x)$ for all $x \in \ell_A$.*

Proof. Let $f \in \ell_A'$. Then $f(x) = \sum_n t_n \sum_k a_{nk} x_k + \sum_k a_k x_k$ for some t, a $\in$ m. Define

$$b_{nk} = 2^{-(n+1)} a_k + t_n a_{nk} \quad \text{for all n and k}.$$

Since

$$\sum_n |b_{nk}| = \sum_n |2^{-(n+1)} a_k + t_n a_{nk}| \leq \sum_n |2^{-(n+1)} a_k| + \sum_n |t_n a_{nk}| \leq H$$

(H independent of k), $B = (b_{nk})$ is an ℓ-ℓ method. Furthermore, for $x \in \ell_A$ we have the relations

$$B(x) = \sum_n \sum_k [2^{-(n+1)} a_k + t_n a_{nk}] x_k = \sum_n \sum_k 2^{-(n+1)} a_k x_k + \sum_n \sum_k t_n a_{nk} x_k$$

$$= \sum_k a_k x_k + \sum_n \sum_k t_n a_{nk} x_k \equiv f(x).$$

Therefore $B \succ A$, and $B(x) = f(x)$ for all $x \in \ell_A$. This completes the proof of the lemma.

We now complete the proof of the necessity portion of the theorem. Let $f \in \ell_A'$ be such that $f(x) = 0$ for $x \in \ell$. By the lemma there exists an ℓ-ℓ method B such that $B \succ A$ and $B(x) = f(x)$ for $x \in \ell_A$. We easily see that if we set $c_{nk} = b_{nk} + a_{nk}$ for all n and k, then $C = (c_{nk})$ is an ℓ-ℓ method and $C(x) = B(x) + A(x)$. So $C \succ A$ and $C \sim A$, since $B(x) = 0$ on ℓ. Thus, by hypothesis, $C(x) = A(x)$ for $x \in \ell_A$, and so $f(x) = 0$ for $x \in \ell_A$. Hence A is perfect. This completes the proof of the theorem.

The theorem of Macphail now becomes a corollary. In order to demonstrate that the results of Theorem 3 extend the theorem of Mcphail, it is only necessary to demonstrate the existence of a perfect nonreversible ℓ-ℓ method that maps into ℓ some sequence not in ℓ. Let $A = (a_{nk})$ be the matrix with all elements in the first row equal to one and all other elements zero. This is an ℓ-ℓ method that maps the sequence $x = \{x_n\}$ into the sequence $\left\{\sum_i x_i, 0, 0, \cdots\right\}$ if x is contained in the space (C) of convergent series. Since A maps into $(0, 0, \cdots)$ every sequence $\{x_n\}$ belonging to (C) that converges to 0, it is not reversible. If $f \in \ell_A'$ and f vanishes on ℓ, then

$$f(x) = \sum_n t_n \sum_k a_{nk} x_k - \sum_k \sum_n t_n a_{nk} x_k \quad \text{for some } t \in m.$$

But if $x \in (C)$, then $f(x) = 0$, and so A is perfect. In this case, $\ell_A = (C)$.

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