

# MINIMAL VARIETIES AND ALMOST HERMITIAN SUBMANIFOLDS

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## 1. INTRODUCTION

In this paper we discuss the geometry of minimal varieties, using a variant of the second fundamental form called the configuration tensor. In Section 2 we define our tensor and prove its basic symmetries; since we use only the properties of affine connections, we are able to deduce the properties of the second fundamental form more quickly than in the classical approach or in the bundle approach of Ambrose [1]. In Section 3 we define minimal variety (our definition is equivalent to the classical one) and give a slight improvement of a theorem of Myers [9] that is useful in Section 7. We then give some formulas (Section 4) for almost Hermitian manifolds; they are needed in Section 5, where we discuss almost Hermitian immersions and prove that an almost Hermitian submanifold of a quasi-Kählerian manifold (usually called a  $*0$ -manifold; see [7], [15]) is a minimal variety. Our theorem provides a simple proof of the well-known fact that a Kähler submanifold is a minimal variety (see [14], [15], [16]), but it also shows, for example, that almost Hermitian submanifolds of  $S^6$  are minimal varieties. We prove this by means of the configuration tensor and the vector cross product in  $R^7$  induced by the Cayley numbers. We also give a necessary and sufficient condition, in terms of the co-derivative of the Kähler form, that an almost Hermitian submanifold of an almost Hermitian manifold be a minimal variety. Quaternionic manifolds are discussed in Section 6; it is shown that a quaternionic submanifold of a quaternionic manifold satisfying a condition analogous to that of a quasi-Kählerian manifold must be totally geodesic. In Section 7 we investigate the definition of minimal surface in  $R^3$  (equivalent to ours) which says that a surface in "conformal representation" is a minimal surface if and only if the coordinate functions are harmonic. We formulate an analogue for arbitrary Riemannian manifolds, and we determine when it is equivalent to our original definition; this leads us to generalize a theorem of Beckenbach and Bing [2].

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## 2. IMMERSIONS

Let  $M$  and  $\bar{M}$  be  $C^\infty$  Riemannian manifolds, with  $M$  immersed in  $\bar{M}$ . Because we shall describe only local properties, we may assume that  $M$  is small enough to be imbedded in  $\bar{M}$  as a proper submanifold. We may then use the following machinery to describe the geometry of  $M$  in  $\bar{M}$ . First we identify  $M$  with a subset of  $\bar{M}$ ; therefore the statement  $M \subset \bar{M}$  shall mean that  $M$  is immersed in  $\bar{M}$ . Let  $\mathfrak{F}(M)$  be the algebra of real-valued differentiable functions on  $M$ , and  $\mathfrak{X}(M)$  the Lie algebra of vector fields on  $M$ , which we may take to be the algebra of derivations of  $\mathfrak{F}(M)$ . Let  $\bar{\mathfrak{X}}(M)$  denote the algebra of restrictions to  $M$  of vector fields of  $\bar{M}$ ; then we may

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write  $\overline{\mathfrak{X}}(M) = \mathfrak{X}(M) \oplus \mathfrak{X}(M)^\perp$ , where  $\mathfrak{X}(M)^\perp$  consists of all vector fields perpendicular to  $M$ . Let  $E: \overline{\mathfrak{X}}(M) \rightarrow \overline{\mathfrak{X}}(M)$  be the identity map, and let  $P: \overline{\mathfrak{X}}(M) \rightarrow \mathfrak{X}(M)$  be the orthogonal projection. We use  $\langle \cdot, \cdot \rangle$  as the metric tensor on  $\overline{\mathfrak{X}}(M)$ ; that is, if  $X, Y \in \overline{\mathfrak{X}}(M)$ , then  $\langle X, Y \rangle \in \mathfrak{F}(M)$ . Let  $\nabla$  be the Riemannian connection of  $M$  determined by the induced metric, and  $\overline{\nabla}$  the Riemannian connection of  $\overline{M}$  restricted to  $\overline{\mathfrak{X}}(M)$ . The *configuration tensor* is the function  $T: \mathfrak{X}(M) \times \overline{\mathfrak{X}}(M) \rightarrow \overline{\mathfrak{X}}(M)$  defined by the formulas

$$(2.1) \quad T_X(Y) = \overline{\nabla}_X(Y) - \nabla_X(Y) \quad \text{for } X, Y \in \mathfrak{X}(M),$$

$$(2.2) \quad T_X(Z) = P\overline{\nabla}_X(Z) \quad \text{for } X \in \mathfrak{X}(M), Z \in \mathfrak{X}(M)^\perp.$$

**PROPOSITION 2.1.** *The configuration tensor has the following properties:*

$$(2.3) \quad T \text{ is bilinear over } \mathfrak{F}(M).$$

$$(2.4) \quad T_X(Y) = T_Y(X) \text{ for } X, Y \in \mathfrak{X}(M).$$

$$(2.5) \quad \langle T_X(Y), Z \rangle = -\langle T_X(Z), Y \rangle \text{ for } X \in \mathfrak{X}(M) \text{ and } Y, Z \in \overline{\mathfrak{X}}(M).$$

$$(2.6) \quad T_X(\mathfrak{X}(M)) \subseteq \mathfrak{X}(M)^\perp \text{ and } T_X(\mathfrak{X}(M)^\perp) \subseteq \mathfrak{X}(M) \text{ for } X \in \mathfrak{X}(M).$$

*Proof.* (2.3) follows from (2.1) and (2.2) by an easy calculation, and (2.4) follows from (2.1) and the fact that

$$\overline{\nabla}_X(Y) - \overline{\nabla}_Y(X) = [X, Y] = \nabla_X(Y) - \nabla_Y(X).$$

Similarly the proofs of (2.5) and (2.6) use the fact that

$$\langle \overline{\nabla}_X(Y), Z \rangle + \langle Y, \overline{\nabla}_X(Z) \rangle = X \langle Y, Z \rangle = \langle \nabla_X(Y), Z \rangle + \langle Y, \nabla_X(Z) \rangle,$$

for  $X, Y, Z \in \mathfrak{X}(M)$ .

We note that (2.5) implies that  $T_X$  is determined by its effect on  $\mathfrak{X}(M)$ .

**PROPOSITION 2.2.** *Let  $X, Y \in \mathfrak{X}(M)$ . Then on  $\mathfrak{X}(M)$  the Gauss equation holds:*

$$(2.7) \quad P\overline{R}_{XY} = R_{XY} - [T_X, T_Y],$$

where  $R_{XY}$  and  $\overline{R}_{XY}$  denote the curvature operators of  $M$  and  $\overline{M}$ . On  $\mathfrak{X}(M)^\perp$  the Codazzi equation holds:

$$(2.8) \quad P\overline{R}_{XY} = T_{[X, Y]} - P[\overline{\nabla}_X, \overline{\nabla}_Y].$$

If on an open subset of  $M$ ,  $\|X\| = \|Y\| = 1$  and  $\langle X, Y \rangle = 0$ , so that they span a field  $\Pi$  of 2-planes, then we may write (2.7) in the form

$$(2.9) \quad K(\Pi) = \langle T_X(X), T_Y(Y) \rangle - \|T_X(Y)\|^2 + \overline{K}(\Pi).$$

*Proof.* On  $\mathfrak{X}(M)$ ,

$$\begin{aligned}
 \overline{PR}_{XY} &= P\overline{\nabla}_{[X,Y]} - P[\overline{\nabla}_X, \overline{\nabla}_Y] \\
 &= \nabla_{[X,Y]} - P\overline{\nabla}_X(\nabla_Y + T_Y) + P\overline{\nabla}_Y(\nabla_X + T_X) \\
 &= \nabla_{[X,Y]} - \nabla_X\nabla_Y - T_X T_Y + \nabla_Y\nabla_X - T_Y T_X \\
 &= R_{XY} - [T_X, T_Y].
 \end{aligned}$$

Similarly, on  $\mathfrak{X}(M)^\perp$

$$\begin{aligned}
 \overline{PR}_{XY} &= P\overline{\nabla}_{[X,Y]} - P[\overline{\nabla}_X, \overline{\nabla}_Y] \\
 &= T_{[X,Y]} - P[\overline{\nabla}_X, \overline{\nabla}_Y].
 \end{aligned}$$

The proof of (2.9) follows easily from (2.7) and the definition of sectional curvature.

If the difference between the dimensions of  $M$  and  $\overline{M}$  is 1, we say that  $M$  is a *hypersurface* of  $\overline{M}$ . We assume that  $M$  is small enough to be orientable so that we can define a unit normal vector field  $N \in \mathfrak{X}(M)^\perp$  globally on  $M$ . In this case we define the notion of *normal curvature*. This is a function  $\kappa$  that assigns to each vector field  $X \in \mathfrak{X}(M)$  a function  $\kappa(X) \in \mathfrak{F}(M)$ , whenever  $X$  is different from zero. It is given by the formula  $\kappa(X) = \|X\|^{-2} \langle T_X(X), N \rangle$ . The *normal curvatures* with respect to a frame field  $\{E_1, \dots, E_n\}$  on  $M$  are given by  $\kappa(E_1), \dots, \kappa(E_n)$ . If  $E_1, \dots, E_n$  are all eigenvectors of the symmetric transformation  $X \rightarrow T_X(N)$ , then  $\kappa(E_1), \dots, \kappa(E_n)$  are called *principal curvatures*.

The relation of the configuration tensor  $T$  to the classical second fundamental form may be described as follows. Let  $\{x_1, \dots, x_{n+k}\}$  be a coordinate system in a neighborhood of  $p \in M$  such that the  $\partial/\partial x_i$  are tangent to  $M$  for  $1 \leq i \leq n$  and the  $\partial/\partial x_\alpha$  are perpendicular to  $M$  for  $n+1 \leq \alpha \leq n+k$ . The second fundamental form  $b_{ij\alpha}$  has three indices  $i, j$ , and  $\alpha$ , which satisfy the conditions  $1 \leq i \leq n$ ,  $1 \leq j \leq n$ , and  $n+1 \leq \alpha \leq n+k$ . It is easy to see that

$$T_{\partial/\partial x_i}(\partial/\partial x_j) = \sum_{\alpha=n+1}^{n+k} b_{ij\alpha} \partial/\partial x_\alpha.$$

It follows from (2.5) that the configuration tensor and the second fundamental form contain the same information.

### 3. MINIMAL VARIETIES

The *mean curvature vector field*  $H$  of  $M \subset \overline{M}$  is defined by

$$H = \sum_{i=1}^n T_{E_i}(E_i),$$

where  $n = \dim M$  and  $\{E_1, \dots, E_n\}$  is an (orthonormal) frame field tangent to  $M$ . The *relative curvature* of  $M \subset \overline{M}$  is the real-valued function  $G$  on  $M$  given by

$$G = \sum_{i,j=1}^n \|T_{E_i}(E_j)\|^2.$$

It is easily verified that these definitions are independent of the choice of the frame field. A submanifold is a *minimal variety* if  $H \equiv 0$ , and it is *totally geodesic* if  $T \equiv 0$ . We call a two-dimensional minimal variety a *minimal surface*. Clearly, a totally geodesic submanifold is a minimal variety, and from (2.5) it follows that a submanifold is totally geodesic if and only if its relative curvature vanishes. We may also speak of a submanifold being a minimal variety or being totally geodesic at a point  $p$ ; this means that  $H = 0$  or  $G = 0$  at  $p$ .

If  $M$  is an orientable hypersurface of  $\bar{M}$  and  $N$  is a unit normal vector field on  $M$ , then

$$\langle H, N \rangle = \sum_{i=1}^n \kappa(E_i).$$

We may choose  $\kappa(E_1), \dots, \kappa(E_n)$  to be the principal curvatures. Thus in the case of a minimal surface in  $R^3$  our definition reduces to the ordinary one.

For an arbitrary Riemannian manifold  $M$  the Ricci curvature and the Ricci scalar curvature are a  $(0, 2)$ -tensor  $k$  and a real-valued function  $R$  given by the formulas

$$k(X, Y) = \sum_{i=1}^n \langle R_{XE_i}(Y), E_i \rangle \quad \text{and} \quad R = \sum_{i,j=1}^n \langle R_{E_i E_j}(E_i), E_j \rangle,$$

where  $\{E_1, \dots, E_n\}$  is any frame field on an open subset of  $M$ , and  $X, Y \in \mathfrak{X}(M)$ . Now assume  $M \subset \bar{M}$ . We define  $\tilde{k}$  and  $\tilde{R}$  by

$$\tilde{k}(X, Y) = \sum_{i=1}^n \langle \bar{R}_{XE_i}(Y), E_i \rangle \quad \text{and} \quad \tilde{R} = \sum_{i,j=1}^n \langle \bar{R}_{E_i E_j}(E_i), E_j \rangle,$$

where  $X, Y \in \mathfrak{X}(M)$  and  $\{E_1, \dots, E_n\}$  is a frame field on an open subset of  $M$ . These may be interpreted as follows. Let  $p \in M$ , and let  $\tilde{M}$  be the submanifold of all  $\bar{M}$  geodesics starting at  $p$  that are initially tangent to  $M$ . Explicitly, we may take

$$\tilde{M} = \overline{\exp_p(U)},$$

where  $U \subset M_p \subset \bar{M}_p$  is a sufficiently small neighborhood of 0 in  $M_p$  and  $\overline{\exp_p}$  denotes the exponential map of  $\bar{M}$ . Although  $\tilde{M}$  may not be totally geodesic everywhere, it is at least totally geodesic at  $p$ . Then  $\tilde{k}$  and  $\tilde{R}$  are respectively the Ricci curvature and the Ricci scalar curvature of  $\tilde{M}$  at  $p$ . Our next theorem, which includes the theorem of Myers [9], relates  $H$  and  $G$  to  $k, \tilde{k}, R,$  and  $\tilde{R}$ .

**THEOREM 3.1.** *Let  $M \subset \bar{M}$  and  $X, Y \in \mathfrak{X}(M)$ . Then*

$$(3.1) \quad k(X, Y) = \tilde{k}(X, Y) + \langle T_X(Y), H \rangle + \text{tr } T_X T_Y,$$

$$(3.2) \quad R = \tilde{R} + \|H\|^2 - G.$$

(Here  $\text{tr}$  denotes the trace.)

*Proof.* We use the Gauss equation to get the relations

$$\begin{aligned}
 k(X, Y) &= \sum_{i=1}^n \left\{ \langle \bar{R}_{XE_i}(Y), E_i \rangle + \langle [T_X, T_{E_i}](Y), E_i \rangle \right\} \\
 &= \tilde{k}(X, Y) + \sum_{i=1}^n \left\{ -\langle T_{E_i}(Y), T_X(E_i) \rangle + \langle T_X(Y), T_{E_i}(E_i) \rangle \right\} \\
 &= \tilde{k}(X, Y) + \text{tr } T_X T_Y + \langle T_X(Y), H \rangle.
 \end{aligned}$$

If we contract once more, we obtain (3.2).

From Theorem 3.1 we obtain the following corollaries.

**COROLLARY 3.2.** *For a minimal variety, we have the formulas*

$$(3.3) \quad k(X, Y) = \tilde{k}(X, Y) + \text{tr } T_X T_Y,$$

$$(3.4) \quad k(X, X) = \tilde{k}(X, X) - \sum_{i=1}^n \|T_X(E_i)\|^2 \leq \tilde{k}(X, X),$$

$$(3.5) \quad R = \tilde{R} - G.$$

**COROLLARY 3.3.** *For a minimal variety,  $R \leq \tilde{R}$ , and equality occurs if and only if  $M$  is totally geodesic.*

**COROLLARY 3.4.** *Suppose  $M \subset \bar{M}$  is a minimal surface. Then the sectional curvatures satisfy the inequality  $K \leq \bar{K}$ , with equality occurring if and only if  $M$  is totally geodesic.*

**COROLLARY 3.5.** *Suppose  $\tilde{R} \leq 0$ ,  $R \geq 0$ , and  $M \subset \bar{M}$  is a minimal variety. Then  $M$  is totally geodesic and  $R = \tilde{R} = 0$ .*

Corollary 3.4 generalizes the fact that a minimal surface in  $R^3$  has nonpositive curvature.

#### 4. COMPLEX STRUCTURE

An *almost complex manifold*  $M$  is a differentiable manifold on which there exists a (1, 1)-tensor  $J$  (which we may regard as an  $\mathfrak{F}(M)$ -linear map  $J: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ ) satisfying the condition  $J^2 = -E$ . Such a manifold is orientable and even-dimensional.  $M$  is an *almost Hermitian manifold* provided it is both almost complex and Riemannian in such a way that  $\langle JX, JY \rangle = \langle X, Y \rangle$  for all  $X, Y \in \mathfrak{X}(M)$ . In the description of the geometry of  $M$ , it is important to consider two special tensors defined in terms of the almost complex structure  $J$ . The first is a 2-form  $F$ , called the *Kähler form*, and it is defined for  $X, Y \in \mathfrak{X}(M)$  by the formula

$$(4.1) \quad F(X, Y) = \langle JX, Y \rangle.$$

Since it is skew-symmetric, it is in fact a differential form. The second, called the *Nijenhuis tensor*, is a (1, 2)-tensor  $S$  defined by

$$(4.2) \quad S(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY],$$

for  $X, Y \in \mathfrak{X}(M)$ . It is easy to show that

$$S(X, Y) = -S(Y, X) \quad \text{and} \quad S(JX, Y) = S(X, JY) = -JS(X, Y).$$

If we extend the Riemannian connection  $\nabla_X$  of  $M$  to be a derivation on the tensor algebra of  $M$ , then we have the formulas

$$(4.3) \quad \nabla_X(J)(Y) = \nabla_X(JY) - J\nabla_X(Y),$$

$$(4.4) \quad \nabla_X(F)(Y, Z) = \langle \nabla_X(J)(Y), Z \rangle.$$

It will be necessary to have explicit formulas for the exterior derivative and co-derivative of the differential form  $F$ . From standard formulas (see Koszul [6]) these can be computed to be

$$(4.5) \quad dF(X, Y, Z) = \mathcal{C}\nabla_X(F)(Y, Z),$$

$$(4.6) \quad \delta F(X) = -\sum_{i=1}^m \{ \nabla_{E_i}(F)(E_i, X) + \nabla_{JE_i}(F)(JE_i, X) \},$$

where  $\mathcal{C}$  denotes the cyclic sum over  $X, Y, Z \in \mathfrak{X}(M)$  and

$$\{ E_1, \dots, E_m, JE_1, \dots, JE_m \}$$

is a frame field on an open subset of  $M$ .

**THEOREM 4.1.** *Let  $X, Y, Z \in \mathfrak{X}(M)$ . Then*

$$(4.7) \quad S(X, Y) = -\nabla_X(J)(JY) + \nabla_{JY}(J)(X) - \nabla_{JX}(J)(Y) + \nabla_Y(J)(JX),$$

$$(4.8) \quad 2\nabla_X(F)(Y, Z) = dF(X, Y, Z) - dF(X, JY, JZ) - \langle X, S(Y, JZ) \rangle,$$

$$(4.9) \quad \begin{cases} 2\nabla_X(F)(Y, Z) + 2\nabla_{JX}(F)(JY, Z) \\ = dF(X, Y, Z) - dF(X, JY, JZ) + dF(Z, JX, JY) + dF(Y, JZ, JX), \end{cases}$$

$$(4.10) \quad \begin{cases} 2\nabla_X(F)(Y, Z) - 2\nabla_{JX}(F)(JY, Z) \\ = \langle S(X, JY), Z \rangle - \langle S(X, Z), JY \rangle - \langle S(JY, Z), X \rangle. \end{cases}$$

*Proof.* The proof of (4.7) follows from the identity  $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$ ; (4.8), (4.9), and (4.10) are consequences of (4.7) and the formula

$$(4.11) \quad \nabla_X(F)(JY, Z) = \nabla_X(F)(Y, JZ).$$

Let  $X, Y \in \mathfrak{X}(M)$ . We shall call an almost Hermitian manifold

*Kählerian* if  $\nabla_X(J) = 0$ ,

*almost Kählerian* if  $dF = 0$ ,

*nearly Kählerian* if  $\nabla_X(J)(Y) + \nabla_Y(J)(X) = 0$ ,

*quasi-Kählerian* if  $\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$ ,

*semi-Kählerian* if  $\delta F = 0$ ,

*Hermitian* if  $S = 0$ .

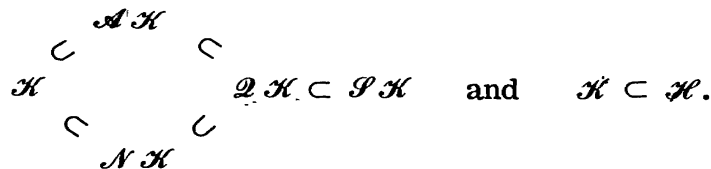
Kotō [7] uses the terms H-space for almost Kählerian, K-space for nearly Kählerian, \*0-space for quasi-Kählerian, and almost semi-Kählerian for semi-Kählerian.

As consequences of Theorem 4.1 we get the following corollaries, which show the relations among the various kinds of almost Hermitian manifolds.

**COROLLARY 4.2.**  $\nabla_X(F)(Y, Z) = \nabla_{JX}(F)(JY, Z)$  if and only if  $M$  is Hermitian.  $\nabla_X(F)(Y, Z) = -\nabla_{JX}(F)(JY, Z)$  if and only if  $M$  is quasi-Kählerian.

Let  $\mathcal{K}$ ,  $\mathcal{AK}$ ,  $\mathcal{NK}$ ,  $\mathcal{QK}$ ,  $\mathcal{PK}$ , and  $\mathcal{H}$  denote the classes of Kähler, almost Kähler, nearly Kähler, quasi-Kähler, semi-Kähler, and Hermitian manifolds, respectively.

**COROLLARY 4.3.** We have the inclusion relations



Furthermore,  $\mathcal{K} = \mathcal{H} \cap \mathcal{QK} = \mathcal{AK} \cap \mathcal{NK}$ .

*Proof.* The relation  $\mathcal{K} \subset \mathcal{AK}$  follows from (4.4) and (4.5),  $\mathcal{AK} \subset \mathcal{QK}$  from (4.4) and (4.9),  $\mathcal{QK} \subset \mathcal{PK}$  from (4.4) and (4.6), and  $\mathcal{K} \subset \mathcal{H}$  from (4.7). It is obvious that  $\mathcal{K} \subset \mathcal{NK}$ , and  $\mathcal{NK} \subset \mathcal{QK}$  is a consequence of (4.11). Furthermore, the relation  $\mathcal{K} \subset \mathcal{H} \cap \mathcal{QK}$  is obvious, and the reverse inclusion follows from (4.9) and (4.10). Finally, if  $M \in \mathcal{NK}$ , then  $dF(X, Y, Z) = 3\nabla_X(F)(Y, Z)$  and hence  $\mathcal{NK} \cap \mathcal{AK} = \mathcal{K}$ .

Fukami and Ishihara [4] have shown that  $S^6$  is nearly Kählerian, and by Corollary 4.3 it is quasi-Kählerian. Since the second Betti number of  $S^6$  is zero,  $S^6$  cannot be Kählerian. More generally, Calabi [3] has shown the existence of an almost complex structure on an orientable hypersurface of  $R^7$  induced by the Cayley numbers. We shall now use the configuration tensor to discuss almost complex structures of this type.

Using the Cayley numbers, we may introduce a vector cross-product  $\times$  that is an  $\mathfrak{F}(R^7)$ -linear map from  $\mathfrak{X}(R^7) \times \mathfrak{X}(R^7)$  to  $\mathfrak{X}(R^7)$  with the following properties (see [3]):

$$(4.12) \quad A \times B = -B \times A,$$

$$(4.13) \quad \langle A \times B, C \rangle = \langle A, B \times C \rangle,$$

$$(4.14) \quad (A \times B) \times C + A \times (B \times C) = 2 \langle A, C \rangle B - \langle B, C \rangle A - \langle A, B \rangle C,$$

$$(4.15) \quad \bar{\nabla}_A(B \times C) = \bar{\nabla}_A(B) \times C + B \times \bar{\nabla}_A(C),$$

for all  $A, B, C \in \mathfrak{X}(R^7)$ ; here  $\bar{\nabla}$  is the Riemannian connection of  $R^7$ .

Let  $M^6$  be an orientable hypersurface of  $R^7$ ; we may choose a unit normal vector field  $N \in \mathfrak{X}(M)^\perp$  globally on  $M$ . We define  $J: \mathfrak{X}(M) \rightarrow \overline{\mathfrak{X}}(M)$  by  $JA = N \times A$ . By (4.12) and (4.13) we see that if  $A \in \mathfrak{X}(M)$ , then  $JA \in \mathfrak{X}(M)$ , and by (4.14) we have the relation  $J^2 A = N \times (N \times A) = -A$ . From this, (4.12), and (4.13), it follows that  $\langle JA, JB \rangle = \langle A, B \rangle$  for  $A, B \in \mathfrak{X}(M)$ . Hence  $M$  is an almost Hermitian manifold with the metric induced from  $R^7$ .

LEMMA 4.4. *Let  $A, B, C \in \mathfrak{X}(M)$ . Then*

$$(4.16) \quad \nabla_A(F)(B, C) = \langle T_A(N), B \times C \rangle,$$

$$(4.17) \quad \nabla_A(F)(B, C) + \nabla_{JA}(F)(JB, C) = \langle T_A(N) + JT_{JA}(N), B \times C \rangle,$$

$$(4.18) \quad \nabla_A(F)(B, C) - \nabla_{JA}(F)(JB, C) = \langle T_A(N) - JT_{JA}(N), B \times C \rangle.$$

Let  $H$  be the mean curvature vector of  $M$  in  $R^7$ , and let  $E_i, E_j, E_k$  be orthonormal vector fields on an open subset of  $M$  such that  $\langle E_i, JE_j \rangle = 0$  and  $E_i \times E_j = E_k$ , where  $E_i, E_j, E_k \in \mathfrak{X}(M)$ . Then  $\{E_1, E_2, E_3, JE_1, JE_2, JE_3\}$  is an orthonormal frame field and

$$(4.19) \quad \begin{aligned} \langle H, N \rangle &= \sum_{i,j,k} \{-\nabla_{E_i}(F)(E_j, E_k) + \nabla_{JE_i}(F)(JE_j, E_k)\} \\ &= -dF(E_i, E_j, E_k) + dF(E_i, JE_j, JE_k) + \nabla_{E_i}(F)(E_j, E_k) \\ &\quad + \nabla_{JE_i}(F)(JE_j, E_k). \end{aligned}$$

*Proof.* For (4.16) we first observe that  $\bar{\nabla}_A(N) = T_A(N)$ . Hence

$$\begin{aligned} \nabla_A(F)(B, C) &= \langle \bar{\nabla}_A(N \times B) - N \times \bar{\nabla}_A(B), C \rangle \\ &= \langle \bar{\nabla}_A(N) \times B, C \rangle \\ &= \langle T_A(N), B \times C \rangle. \end{aligned}$$

The proofs of (4.17) and (4.18) are obvious consequences of (4.16). For (4.19), we note that

$$\begin{aligned} \langle H, N \rangle &= \left\langle \sum_{i,j,k} \{T_{E_i}(E_i) + T_{JE_i}(JE_i)\}, N \right\rangle \\ &= \sum \left\{ -\langle T_{E_i}(N), E_j \times E_k \rangle + \langle T_{JE_i}(N), JE_j \times E_k \rangle \right\} \\ &= \sum \{-\nabla_{E_i}(F)(E_j, E_k) + \nabla_{JE_i}(F)(JE_j, E_k)\}. \end{aligned}$$

We now state Lemma 4.4 in the following form, due to Calabi [3]:

THEOREM 4.5. *Let  $M^6$  be an orientable hypersurface of  $R^7$  with the almost complex structure induced by the Cayley numbers. Then*

$$(4.20) \quad M \text{ is Kählerian if and only if it is totally geodesic;}$$

$$(4.21) \quad M \text{ is quasi-Kählerian if and only if } T_A(N) + JT_{JA}(N) = 0 \text{ for all } A \in \mathfrak{X}(M);$$



(4.22)  $M$  is Hermitian if and only if  $T_A(N) - JT_{JA}(N) = 0$ ;

(4.23) if  $M$  is either almost Kählerian or Hermitian, it is a minimal variety of  $R^7$ .

We may now prove very simply that  $S^6$  (imbedded as the unit sphere in  $R^7$ ) is nearly Kählerian; we merely observe that  $T_A(N) = A$  for  $A \in \mathfrak{X}(S^6)$ , and use (4.13). Similarly,  $S^2 \times R^4$  is quasi-Kählerian but not almost Kählerian or nearly Kählerian. This follows from the fact that  $T_A(N) = A_1$  for  $A \in \mathfrak{X}(S^2 \times R^4)$ , where  $A_1$  is the component of  $A$  tangent to  $S^2$ . A calculation then shows that  $S^2 \times R^4$  has the stated properties. Note that the almost complex structure on  $S^2 \times R^4$  under consideration is not the usual one.

As Calabi [3] has remarked, the problem of finding an almost Kähler manifold that is not Kählerian seems difficult. The present author has been unable to find an example.

### 5. ALMOST HERMITIAN IMMERSIONS

Let  $M$  and  $\bar{M}$  be almost Hermitian manifolds with  $M \subset \bar{M}$ . We say that  $M$  is an *almost Hermitian submanifold* of  $\bar{M}$  if  $JX \in \mathfrak{X}(M)$  whenever  $X \in \mathfrak{X}(M)$ , where  $J$  is the almost complex structure of  $\bar{M}$ . Thus the almost complex structure of  $M$  is the restriction to  $\mathfrak{X}(M)$  of the almost complex structure of  $\bar{M}$ , and we denote both by  $J$ .

**LEMMA 5.1.** *Let  $S$  and  $F$  be the Nijenhuis tensor and the Kähler form of  $\bar{M}$ , respectively. Then the restrictions of  $S$ ,  $F$ , and  $dF$  to  $\mathfrak{X}(M)$  are the Nijenhuis tensor, the Kähler form, and the derivative of the Kähler form of  $M$ .*

*Proof.* Since the bracket and almost complex structure of  $M$  are simply the restrictions of the corresponding objects of  $\bar{M}$ , the Nijenhuis tensor of  $M$  is the restriction of  $S$ . Similarly, the Kähler form of  $M$  is the restriction of  $F$ , since the metric tensor is the induced one. Since  $d$  commutes with the inclusion map,  $dF$  also restricts to  $\mathfrak{X}(M)$  to give the derivative of the Kähler form of  $M$ .

Henceforth, we let  $S$  stand for the Nijenhuis tensor and  $F$  for the Kähler form of both  $M$  and  $\bar{M}$ .

**PROPOSITION 5.2.** *If  $\bar{M}$  is Kählerian, almost Kählerian, nearly Kählerian, quasi-Kählerian, or Hermitian, then any almost Hermitian submanifold of  $\bar{M}$  has the same property.*

*Proof.* For the classes  $\mathcal{AK}$  and  $\mathcal{K}$  the proof follows directly from the preceding lemma. For the classes  $\mathcal{K}$  and  $\mathcal{QK}$ , the proof uses the preceding lemma together with (4.8) and (4.9), respectively. That the  $\mathcal{NK}$ -property is inherited can be proved by a direct calculation.

We shall use terminology such as “ $M$  is an almost Kähler submanifold of  $\bar{M}$ ” to mean that  $\bar{M}$  is almost Kählerian and  $M$  is an almost Hermitian submanifold.

**PROPOSITION 5.3.** *If  $M \subset \bar{M}$  is a Kähler submanifold and  $X, Y \in \mathfrak{X}(M)$ ,  $Z \in \mathfrak{X}(M)^\perp$ , then*

$$(5.1) \quad T_X(JY) = JT_X(Y); \quad T_X(JZ) = JT_X(Z),$$

$$(5.2) \quad T_{JX}(Z) = -JT_X(Z).$$

*Proof.* We have the relations

$$T_X(JY) = \bar{\nabla}_X(JY) - \nabla_X(JY) = J(\bar{\nabla}_X(Y) - \nabla_X(Y)) = JT_X(Y).$$

From this, (2.4), and (2.5), the rest of the proposition follows easily.

The fact that  $J$  and  $T_X$  commute determines to a large extent the behavior of  $T_X$ . For example, it is easy to show that if  $M^2$  is a Kähler submanifold of  $\bar{M}^{2+k}$ , then  $T_X^2(Y) = \frac{1}{2}(K - \bar{K}) \|X\|^2 Y$  for  $X, Y \in \mathfrak{X}(M)$ . Another application is to isotropic immersions (see O'Neill [12]).

**PROPOSITION 5.4.** *If  $M \subset \bar{M}$  is a quasi-Kähler submanifold, then*

$$(5.3) \quad T_X(Y) + T_{JX}(JY) = 0 \text{ for all } X, Y \in \mathfrak{X}(M).$$

*If  $M \subset \bar{M}$  is a Hermitian submanifold, then*

$$(5.4) \quad T_X(Y) - T_{JX}(JY) = -J(T_X(JY) + T_{JX}(Y)).$$

*Proof.* For (5.3), we note that

$$\begin{aligned} 0 &= \bar{\nabla}_X(J)(X) + \bar{\nabla}_{JX}(J)(JX) \\ &= \nabla_X(J)(X) + \nabla_{JX}(J)(JX) + T_X(JX) - JT_X(X) - T_{JX}(X) - JT_{JX}(JX) \\ &= -J(T_X(X) + T_{JX}(JX)). \end{aligned}$$

Since the right-hand side is symmetric in  $X$  and  $Y$ , it follows that

$$T_X(Y) + T_{JX}(JY) = 0.$$

The proof of (5.4) is similar.

Our next task is to investigate the relation between minimal varieties and almost Hermitian submanifolds. First note that the co-derivative  $\delta F$  of  $F$  on  $M$  is not generally the same as the co-derivative  $\bar{\delta} F$  of  $F$  on  $\bar{M}$ . It will be useful to consider yet another "co-derivative"  $\tilde{\delta} F$ , defined by the formula

$$(5.5) \quad \tilde{\delta} F(X) = - \sum_{i=1}^m \{ \bar{\nabla}_{E_i}(F)(E_i, X) + \bar{\nabla}_{JE_i}(F)(JE_i, X) \},$$

where  $\{E_1, \dots, E_m, JE_1, \dots, JE_m\}$  is a frame field on an open subset of  $M$  and  $X \in \mathfrak{X}(M)$ . On  $\mathfrak{X}(M)$  we may interpret  $F$  similarly to  $k$  and  $\bar{R}$ . First note that we may restrict  $F$  to a differential form on any submanifold of  $\bar{M}$ , whether it is almost Hermitian or not. Let  $p \in M$ , and let  $\tilde{M}$  be the manifold consisting of all  $\bar{M}$  geodesics starting at  $p$  and tangent to  $M$ . Then we may interpret  $\tilde{\delta}$  on  $\mathfrak{X}(M)$  as the co-derivative at  $p$  with respect to this submanifold. Even if  $\tilde{M}$  is not almost Hermitian, we may still consider  $\tilde{\delta} F$ , which is given by (5.5). However, it will be more useful to consider  $\bar{\delta} F$  on  $\mathfrak{X}(M)^\perp$ .

**PROPOSITION 5.5.** *Let  $Z \in \mathfrak{X}(M)^\perp$ . Then*

$$(5.6) \quad \tilde{\delta} F(Z) = \langle JH, Z \rangle.$$

*Proof.* We have the relations

$$\begin{aligned} \tilde{\delta} F(Z) &= - \sum_{i=1}^m \left\{ \langle \bar{\nabla}_{E_i}(J)(E_i), Z \rangle + \langle \bar{\nabla}_{JE_i}(J)(JE_i), Z \rangle \right\} \\ &= - \sum_{i=1}^m \langle T_{E_i}(JE_i) - JT_{E_i}(E_i) - T_{JE_i}(E_i) - JT_{E_i}(E_i), Z \rangle \\ &= \langle JH, Z \rangle. \end{aligned}$$

Proposition 5.5 immediately leads to the following result.

**THEOREM 5.6.** *An almost Hermitian submanifold  $M \subset \bar{M}$  is a minimal variety if and only if  $\tilde{\delta} F = 0$  on  $\mathfrak{X}(M)^\perp$ .*

**THEOREM 5.7.** *A quasi-Kähler submanifold is a minimal variety.*

*Proof.* By (5.3),

$$H = \sum_{i=1}^m \{ T_{E_i}(E_i) + T_{JE_i}(JE_i) \} = 0.$$

We note one further property of quasi-Kähler submanifolds. The *holomorphic curvature*  $K_h$  of an almost Hermitian manifold is a function that assigns to each  $X \in \mathfrak{X}(M)$  the sectional curvature of the field of 2-planes spanned by  $X$  and  $JX$  (whenever  $X$  is nonzero).

**PROPOSITION 5.8.** *The holomorphic curvature  $K_h$  of a quasi-Kähler submanifold is not greater than the holomorphic curvature  $\bar{K}_h$  of  $\bar{M}$ .*

*Proof.* From (2.9) it follows that if  $X \in \mathfrak{X}(M)$ , then

$$(5.7) \quad (K_h(X) - \bar{K}_h(X)) \|X\|^4 = - \|T_X(X)\|^2 - \|T_X(JX)\|^2$$

wherever  $X$  is different from zero.

We remark that the work of O'Neill [11] and Ôtsuki [13] on compact minimal varieties in complete Riemannian manifolds applies *a fortiori* to compact quasi-Kähler submanifolds.

## 6. QUATERNIONIC STRUCTURE

Analogously to the way that complex numbers suggest various complex structures, we can use the quaternions to form quaternionic structures on a differentiable manifold. It turns out that the additional structure tends to make the various geometrical relations trivial.

A differentiable manifold is *almost quaternionic* if it has two almost complex structures  $I$  and  $J$  satisfying the condition  $IJ + JI = 0$ . If  $M$  is a Riemannian manifold in which both  $I$  and  $J$  are isometries, we say that  $M$  is *q-almost Hermitian*. Similarly, if  $(M, I)$  and  $(M, J)$  are both in one of the classes that we have defined, then we indicate this by prefixing a "q" to the class of  $M$ .

Obata [10] has proved that the Ricci curvature of a  $q$ -Kähler manifold vanishes. This says, for example, that quaternionic projective space is not  $q$ -Kählerian, and in fact Massey [8] has shown that it is not even almost complex. It would be interesting to find some nonflat examples of  $q$ -Kähler manifolds. The following theorem about quaternionic submanifolds again demonstrates the simple nature of quaternionic structure.

**THEOREM 6.1.** *Let  $M$  be a  $q$ -quasi-Kähler submanifold of  $\bar{M}$ ; that is, assume that any of the almost complex structures  $I, J, IJ$  makes  $\bar{M}$  quasi-Kählerian, and that  $M$  is almost Hermitian with respect to  $I, J$ , and  $IJ$ . Then  $M$  is totally geodesic.*

*Proof.* It is clear that  $M$  itself is  $q$ -quasi-Kählerian. Furthermore,

$$0 = T_X(Y) + T_{IX}(IY) = T_X(Y) + T_{JX}(JY) = T_{IX}(IY) + T_{JX}(JY).$$

Hence  $T_X(Y) = 0$  for all  $X, Y \in \mathfrak{X}(M)$ .

### 7. CONFORMAL STRUCTURE

A classical theorem of Weierstrass states that a surface in  $R^3$  in “conformal representation” is a minimal variety if and only if the coordinate functions are harmonic. In this section we discuss how this theorem can be generalized to an arbitrary Riemannian manifold. For this purpose we need some information about conformal equivalence, and in particular about conformal flatness.

*Definition.* Let  $(M, \langle \cdot, \cdot \rangle)$  and  $(M^0, \langle \cdot, \cdot \rangle^0)$  be two Riemannian manifolds. Then  $M$  and  $M^0$  are *conformally equivalent* if and only if there exists a diffeomorphism  $\phi: M \rightarrow M^0$  and a differentiable function  $\sigma: M \rightarrow R$  such that

$$(7.1) \quad \langle X^0, Y^0 \rangle = (e^{2\sigma} \langle X, Y \rangle) \circ \phi^{-1},$$

where  $X, Y \in \mathfrak{X}(M)$  and  $X^0, Y^0 \in \mathfrak{X}(M^0)$  are the vector fields corresponding to  $X$  and  $Y$  induced on  $M^0$  by  $\phi$ . If  $M$  is flat, we say that  $M^0$  is *conformally flat*.

**LEMMA 7.1.** *Suppose  $M$  and  $M^0$  are conformally equivalent. Let  $\nabla$  and  $\nabla^0$  denote the corresponding Riemannian connections. If  $X, Y \in \mathfrak{X}(M)$ , then*

$$(7.2) \quad \nabla_{X^0}^0(Y^0) = (\nabla_X(Y) + X(\sigma)Y + Y(\sigma)X - \langle X, Y \rangle \text{grad } \sigma)^0.$$

*Proof.* This follows from (7.1) and the formula

$$2 \langle \nabla_X(Y), Z \rangle = X \langle Y, Z \rangle - \langle X, [Y, Z] \rangle + Y \langle X, Z \rangle - \langle Y, [X, Z] \rangle - Z \langle X, Y \rangle + \langle Z, [X, Y] \rangle.$$

We now generalize the Weierstrass theorem. For this purpose we consider conformally equivalent Riemannian manifolds  $M$  and  $M^0$ , and we suppose  $M^0$  is immersed in  $\bar{M}$ . Let  $\{E_1, \dots, E_n\}$  be a frame field on an open subset of  $M$ . Our generalization of the Laplacian will be  $\sum_{i=1}^n \bar{\nabla}_{E_i^0}(E_i^0)$ ; if  $\bar{M}$  is some Euclidean space, then this expression reduces to the ordinary Laplacian, modulo the usual canonical identifications. First we need some more lemmas.

**LEMMA 7.2.** *We have the formula*

$$(7.3) \quad \sum_{i=1}^n \nabla_{E_i}^0(E_i^0) = \left( \sum_{i=1}^n \nabla_{E_i}(E_i) - (n-2)(\text{grad } \sigma) \right)^0.$$

*Proof.* This is easily calculated from (7.2) and the fact that

$$\text{grad } \sigma = \sum_{i=1}^n E_i(\sigma) E_i.$$

**COROLLARY 7.3.** *Suppose  $M^0$  is conformally flat and  $\{E_1, \dots, E_n\}$  are the natural coordinate vector fields of Euclidean space.*

*If  $\dim M^0 \geq 3$  and  $\sum_{i=1}^n \nabla_{E_i}^0(E_i^0) = 0$ , then  $M^0$  is flat.*

*If  $\dim M^0 = 2$ , then  $\nabla_{E_1}^0(E_1^0) + \nabla_{E_2}^0(E_2^0) = 0$ .*

*Proof.* In the case at hand,  $\nabla_{E_i}(E_i) = 0$  for  $i = 1, \dots, n$ . If  $\dim M^0 \geq 3$  and  $\sum_{i=1}^n \nabla_{E_i}^0(E_i^0) = 0$ , then by (7.3)  $\text{grad } \sigma = 0$  and so  $M$  is flat. The conclusion for  $\dim M^0 = 2$  is also obvious from (7.3).

**LEMMA 7.4.** *Suppose  $M^0$  is a conformally flat manifold of dimension  $n$  immersed in  $\bar{M}$ . If  $\{E_1, \dots, E_n\}$  are the natural coordinate vector fields of Euclidean space, then*

$$(7.4) \quad \sum_{i=1}^n \bar{\nabla}_{E_i}^0(E_i^0) = e^{2\sigma} H - (n-2)(\text{grad } \sigma)^0.$$

*Proof.* It is sufficient to observe that

$$\begin{aligned} \sum_{i=1}^n \bar{\nabla}_{E_i}^0(E_i^0) &= \sum_{i=1}^n \{ \nabla_{E_i}^0(E_i^0) + T_{E_i^0}(E_i^0) \} \\ &= \sum_{i=1}^n \{ (\nabla_{E_i}(E_i))^0 + T_{E_i^0}(E_i^0) \} - (n-2)(\text{grad } \sigma)^0 \\ &= e^{2\sigma} H - (n-2)(\text{grad } \sigma)^0. \end{aligned}$$

Now we can prove the main theorem.

**THEOREM 7.5.** *Suppose  $M^0$  is a conformally flat submanifold of  $\bar{M}$  of dimension  $n$ , and let  $\{E_1, \dots, E_n\}$  be the natural coordinate vector fields of Euclidean space.*

(i) *If  $\dim M^0 = 2$ , then  $M^0$  is a minimal variety of  $\bar{M}$  if and only if*

$$\bar{\nabla}_{E_1}^0(E_1^0) + \bar{\nabla}_{E_2}^0(E_2^0) = 0.$$

(ii) If  $\dim M^0 \geq 3$ ,  $\tilde{R} \leq 0$ , and  $\sum_{i=1}^n \bar{\nabla}_{E_i^0}(E_i^0) = 0$ , then  $M^0$  is a flat totally geodesic submanifold of  $\bar{M}$ .

*Proof.* If  $\dim M^0 = 2$ , then (7.4) reduces to the relation

$$\bar{\nabla}_{E_1^0}(E_1^0) + \bar{\nabla}_{E_2^0}(E_2^0) = e^{2\sigma} H,$$

and (i) follows. If  $\dim M^0 \geq 3$  and  $\sum_{i=1}^n \bar{\nabla}_{E_i^0}(E_i^0) = 0$ , then (7.4) becomes  $e^{2\sigma} H - (n - 2)(\text{grad } \sigma)^0 = 0$ . Now  $H$  is perpendicular to  $M^0$ , while  $(\text{grad } \sigma)^0$  is tangent. Hence  $H = 0$  and  $\sigma$  is constant. The latter condition implies that  $M^0$  is flat; hence its Ricci scalar curvature  $R$  vanishes. Therefore, by Corollary 3.5 and the hypothesis  $\tilde{R} \leq 0$ , we conclude that  $M^0$  is totally geodesic.

#### REFERENCES

1. W. Ambrose, *The Cartan structural equations in classical Riemannian geometry*, J. Indian Math. Soc. (N.S.) 24 (1960), 23-76.
2. E. F. Beckenbach and R. H. Bing, *Conformal minimal varieties*, Duke Math. J. 10 (1943), 637-640.
3. E. Calabi, *Construction and properties of some 6-dimensional almost complex manifolds*, Trans. Amer. Math. Soc. 87 (1958), 407-438.
4. T. Fukami and S. Ishihara, *Almost Hermitian structure on  $S^6$* , Tôhoku Math. J. (2) 7 (1955), 151-156.
5. K. Kommerell, *Riemannsche Flächen im ebenen Raum von vier Dimensionen*, Math. Ann. 60 (1905), 548-596.
6. J. L. Koszul, *Variétés Kahleriennes*, Notes, São Paulo, 1957.
7. S. Kotô, *Some theorems on almost Kählerian spaces*, J. Math. Soc. Japan, 12 (1960), 422-433.
8. W. S. Massey, *Non-existence of almost-complex structures on quaternionic projective spaces*, Pacific J. Math., 12 (1962), 1379-1384.
9. S. B. Myers, *Curvature of closed hypersurfaces and nonexistence of closed minimal hypersurfaces*, Trans. Amer. Math. Soc. 71 (1951), 211-217.
10. M. Obata, *Affine connections on manifolds with almost complex, quaternion, or Hermitian structure*, Jap. J. Math. 26 (1956), 43-77.
11. B. O'Neill, *Immersion of manifolds of nonpositive curvature*, Proc. Amer. Math. Soc. 11 (1960), 132-134.
12. ———, *Isotropic and Kähler immersions*, Canad. J. Math. (to appear).
13. T. Ôtsuki, *Isometric imbedding of Riemann manifolds in a Riemann manifold*, J. Math. Soc. Japan 6 (1954), 221-234.
14. G. de Rham, *On the area of complex manifolds*, Notes, Institute for Advanced Study, Princeton, 1957.
15. J. A. Schouten and K. Yano, *On invariant subspaces in the almost complex  $X_{2n}$* , Indag. Math. 17 (1955), 261-269.

16. W. Wirtinger, *Eine Determinantenidentität und ihre Anwendung auf analytische Gebilde in euklidischer und Hermitescher Massbestimmung*, *Monatsh. Math. Phys.* 44 (1936), 343-365.

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