ON THE THOM CLASS OF A SUBMANIFOLD

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To H. Hopf on his 70th birthday, with admiration and gratitude.

1. INTRODUCTION

Let M and N be compact, connected, oriented topological manifolds, of dimensions m and n, with N contained in M (a subspace of M) and with $n < m$; let μ and ν be fundamental cycles of M and N (see [6]), that is, generators of the (singular) homology groups $H_m(M)$ and $H_n(N)$. It is our purpose (a) to indicate a simple construction of the Thom class $u = u_N \in H^{m-n}(M, M - N)$ and the Thom isomorphism $H^s(N) \approx H^{s+m-n}(M, M - N)$ [essentially given by cup-product with u], together with a proof that the Thom class, made absolute, is Poincaré-dual to the image of ν in $H_n(M)$ (see [7], where all this originated), and (b) to prove for two submanifolds of M of dual dimensions, whose intersection is a finite set, that the intersection number (defined *via* cohomology, as usual) is what intuitively it should be, namely the sum of the indices of the intersection points (a "well-known fact"). Our basic tool is a lemma of V. Klee concerning local flatness [3].

2. THOM CLASS

Homology means singular homology with integral coefficients. We write D^q and S^{q-1} for unit-disk and unit-sphere in real q-space R^q; we let d^q and δ_q denote the standard generators of

$$H^q(R^q, R^q - \{0\}) \quad \text{and} \quad H_q(R^q, R^q - \{0\}).$$

We write ε for the homology class defined by a point with multiplicity 1. With m and n as in Section 1, we shall throughout identify R^n, respectively, R^{m-n}, with the subspaces of R^m spanned by the first n, respectively, the last $m - n$, axes. We recall the definitions (a) of N being locally flat in M: each point p of N has a ("flat") neighborhood T such that the pair $(T, T \cap N)$ is homeomorphic to the pair (R^m, R^n), and (b) of N having a normal bundle neighborhood in M: a neighborhood V and a retraction $r: V \to N$ such that each point p of N has a neighborhood W in N with $r: r^{-1}(W) \to W$ equivalent to $p_1: W \times R^{m-n} \to W$.

To construct the Thom class, we first assume N locally flat in M. Let U be an open set in N, whose closure is contained in a "flat" neighborhood T (see (a) above). The relative cohomology group $H^s((M - N) \cup U, M - N)$ is isomorphic, by excision, first to $H^s((R^m - R^n) \cup U, R^m - R^n)$ then to

$$H^s(U \times R^{m-n}, U \times (R^{m-n} - \{0\})) = H^s(U \times (R^{m-n}, R^{m-n} - \{0\})),$$

and finally to $H^{s-m+n}(U)$, by the correspondence $x \leftrightarrow x \times d^{m-n}$ (Künneth Theorem). It is therefore 0 for $s < m - n$; and for $s = m - n$ there is the class u_U corresponding to $1 \in H^0(U)$ and characterized by the fact that $\langle u_U, \gamma \rangle = 1$ for any

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"transversal" (m - n)-cell, in other words, for any $\gamma \in H_{m-n}((M - N) \cup U, M - N)$ that corresponds to $\varepsilon \times \delta_{m-n}$ in $H_{m-n}(U \times (R^{m-n}, R^{m-n} - \{0\}))$.

We now build up the Thom class u in $H^{m-n}(M, M - N)$ (characterized by $\langle u, \gamma \rangle = 1$ for any γ as above, at any point of N) and prove at the same time that $H^s(M, M - N) = 0$ for $s < m - n$, by the standard procedure of covering N with a finite number of U's, adjoining one at a time and using the Mayer-Vietoris sequence.

We now turn to the general case. A lemma of Klee [3] shows, by a simple construction, that N (= N × 0) is always locally flat in $M \times D^n \overset{\text{def}}{=} P$; thus there is a Thom class u' in P, of dimension m (it makes no difference for the proof that P has a boundary). We now note that the pair (P, P - N) can be written as $(M, M - N) \times (D^n, D^n - \{0\})$. The correspondence $x \leftrightarrow x \times d^n$ is therefore an isomorphism, say κ, of $H^*(M, M - N)$ and $H^*(P, P - N)$. We define the Thom class u as the class with $u \times d^n = u'$. This is consistent with the earlier definition, in case N happens to be locally flat.

3. THOM ISOMORPHISM

Let V be a neighborhood of N in M, with inclusion map $k: V \subset M$, that retracts onto N; let r be a retraction (for existence of V and r see [4, 12.2a and 19.2]). Let e^* be the excision map $H^*(V, V - N) \approx H^*(M, M - N)$.

THEOREM A. *The map* $\phi^*: H^*(N) \to H^*(M, M - N)$, *defined by*

$$\phi^* x = e^*(r^* x \cdot k^* u) ,$$

is an isomorphism; it is independent of the choice of V *and* r.

(There is a similar isomorphism in homology.)

We first show that the choice of r does not matter: Let V' be a second neighborhood of N, contained in V, with retraction $r': V' \to N$, such that r' is homotopic to the identity of V' *within* V; this exists, of course. By a little diagram-chasing one sees that $H^*(V, V')$ maps onto $H^*(V, N)$. For a given $x \in H^*(N)$, let y be any element of $H^*(V)$ that restricts to x (as does r^*x). Then $y - r^*x$ pulls back into $H^*(V, N)$ and therefore also into $H^*(V, V')$. The product with k^*u lies in $H^*(V, V' \cup (V - N)) = 0$.

Next, it is clear from naturality that the choice of V also doesn't matter; thus ϕ^* is well-defined.

To prove ϕ^* an isomorphism, we work in P (so that N is locally flat), with u' in place of u; V is a neighborhood of N in P. Let U be a small open set in N, as in Section 2. We can then replace $H^*((P - N) \cup U, P - N)$ by

$$H^*(U \times R^m, U \times (R^m - \{0\})),$$

and the latter by $H^*(W, W - U)$, where W is the (open) subset of $U \times R^m$ corresponding to $r^{-1}(U)$. Clearly, the Thom class u' goes over into the class represented by the generator d^m of $H^m(R^m, R^m - \{0\})$, and the map ϕ^*, transferred to W, gives an isomorphism of $H^*(U)$ with $H^*(W, W - U)$ (by excision and the Künneth theorem; we note that, as pointed out above, we can in the definition of ϕ^* replace the retraction $r = r \mid W$ of W onto U by the ordinary projection of $W \subset U \times R^m$ onto U). Thus ϕ^* gives an isomorphism from $H^*(U)$ to $H^*((P - N) \cup U, P - N)$. Covering N by a finite number of U's and repeatedly applying the Mayer-Vietoris sequence yields Theorem A for P. The result for M follows immediately from the isomorphism $x \leftrightarrow x \times d^n$ of Section 2.

[A shorter, less elementary approach to all this goes as follows: The results of Curtis-Lashof and Milnor [1], [5] and of Kister [2] show that N has a normal bundle neighborhood in $M \times D^q$ for large q. We transfer the standard Thom class and isomorphism for this bundle back to M by an isomorphism like κ above.]

4. POINCARÉ DUALITY

THEOREM B. *Let* M, N, μ, ν *be as in Section* 1, *and let* u *be the Thom class of* N *in* M; *then the image of* ν *in* $H_n(M)$ *is the Poincaré-dual to* $\tilde{u}$, *the image of* u *in* $H^{m-n}(M)$: $\mathfrak{d}\tilde{u} = i_* \nu$. (Here $i: N \subset M$ and $\mathfrak{d} = \cap \mu$.)

Proof. We first assume that N is locally flat in M. Let V, $k: V \subset M$, and $r: V \to N$ be as in the beginning of Section 3, and let r be homotopic to the identity map of V within M. Let $\tilde{\mu}$ be the element of $H_n(M, M - N)$ determined by μ; then

$$\mathfrak{d}\tilde{u} = \tilde{u} \cap \mu = u \cap \mu = u \cap \tilde{\mu}$$

[$H^*(M, M - N)$ and $H_*(M, M - N)$ are paired to $H_*(M)$]. From naturality it is clear that $u \cap \tilde{\mu}$ is k_*-image of an element of $H_n(V)$. Our homotopy assumption on r implies that $k_* = i_* \cdot r_*$, and thus $u \cap \tilde{\mu}$ is i_*-image of some element of $H_n(N)$; in other words, $u \cap \mu = t \cdot i_* \nu$ for some integer t. We must show that $t = 1$; we do this by a local argument.

Let p be a point of N, and let T be a "flat" neighborhood of p; that is, let $(T, T \cap N) \approx (R^m, R^n)$, with p going to 0. The Thom class u turns into the Thom class u_0 of R^n in R^m. The cycle μ yields, *via* excision, the fundamental cycle μ_0 of R^m mod $R^m - \{0\}$, and similarly ν yields the fundamental cycle ν_0 of R^n mod $R^n - \{0\}$, or equivalently of R^m mod $R^m - R^{m-n}$. Naturality and the elementary relation $u_0 \cap \mu_0 = \nu_0$ imply $t = 1$. [In more detail: We may assume that V near p is just T; let Q be the set in T corresponding to R^{m-n}. We interpret u in $H^{m-n}(V, V - N)$, μ in $H_m(V, V - \{p\})$, with $u \cap \mu$ in $H_n(V, V - Q)$ (because $V - \{p\} = (V - N) \cup (V - Q)$). By excision of V - T, the last two groups are equal to $H_m(R^m, R^m - \{0\})$ and $H_n(R^m, R^m - R^{m-n})$. We then apply the permanence relation for $g: T \subset V$ in the form $g_*(g^* u \cap \mu_0) = u \cap g_* \mu_0$, with $g^* u = u_0$. Thus $i_* \nu = g_* \nu_0 = g_*(u_0 \cap \mu_0) = u \cap g_* \mu_0 = u \cap \mu = t \cdot i_* \nu$ (in $H_n(V, V - Q)$, and so $t = 1$].

For the general case we go to $P = M \times D^n$. We have the relations $\tilde{\mu}_P = \tilde{\mu} \times \delta_n$, $u' = u \times d^n$, $d^n \cap \delta_n = \varepsilon$, and

$$u' \cap \tilde{\mu}_P = (u \cap \tilde{\mu}) \times (d^n \cap \delta_n) = (u \cap \tilde{\mu}) \times \varepsilon;$$

and it follows that $u \cap \tilde{\mu} = i_* \nu$. (The fact that P has a boundary introduces no complications, since u' exists on P.)

5. INTERSECTION NUMBER

Let A and B be two submanifolds (compact, connected, oriented) of M, of dual dimensions n and $m - n = q$, with fundamental cycles α and β. The intersection number $A \circ B$ is defined as

$$\langle \mathfrak{d}^{-1} i_* \alpha \cup \mathfrak{d}^{-1} j_* \beta, \mu \rangle ;$$

here i, j are the respective inclusions of A, B in M. Suppose now the intersection $A \cap B$ is finite. With each p of $A \cap B$ one associates an index or a local intersection number j_p as follows: Let V be a small neighborhood of p, homeomorphic to R^m, and let A_0 be the component of $V \cap A$ containing p. Then $H_{q-1}(V - A_0)$ is infinite cyclic (by Alexander duality), with a preferred generator σ determined by the orientations of A_0 (from that of A) and of V (from that of M). (We could again use the device of making A locally flat, by multiplying with D^n; the structure of $H_{q-1}(V - A_0)$ follows then from the Künneth theorem.) A small (q - 1)-sphere around p in B, with orientation induced from B, yields the element $j_p \sigma$ of $H_{q-1}(V - A_0)$. It is clear that j_p is independent of the choices made.

THEOREM C. $A \circ B = \sum_{p \in A \cap B} j_p$; *that is, the (global) intersection number is the sum of the local intersection numbers.*

Proof. We write u and v for the Thom classes of A and B, and $\tilde{u}$, $\tilde{v}$ for the corresponding absolute elements, so that

$$A \circ B = \langle \tilde{u} \cup \tilde{v}, \mu \rangle = \langle \tilde{u}, \tilde{v} \cap \mu \rangle = \langle \tilde{u}, i_* \beta \rangle ,$$

by Theorem B; writing β' for the element of $H_q(M, M - A)$ determined by $i_* \beta$, we have thus the relation $A \circ B = \langle u, \beta' \rangle$. We assume now first that A is locally flat, and take a small neighborhood V of A in M such that near each $p \in A \cap B$ the pair (V, A) looks like (R^m, R^n). Clearly, the relative cycle of R^m mod $R^m - R^n$ determined by β', *via* excision, is $j_p \gamma_p$, where γ_p corresponds to the generator of $H_q(R^{m-n}, R^{m-n} - \{0\})$. Thus the element of $H_q(V, V - A)$ determined by β' is $\sum j_p \gamma_p$. The Thom class u is represented by a class $\bar{u} \in H^q(V, V - A)$ with $\langle \bar{u}, \gamma_p \rangle = 1$ for all $p \in A \cap B$. Thus

$$A \circ B = \langle u, \beta' \rangle = \langle \bar{u}, \sum j_p \gamma_p \rangle = \sum j_p .$$

For the general case we use again $P = M \times D^n$, in which $A = A \times 0$ is locally flat, and replace B by $B \times D^n$ (fundamental cycle $\beta \times \delta_n$). Then

$$A \circ B = \langle \tilde{u} \cup \tilde{v}, \mu \rangle = \langle \tilde{u} \times 1 \cup \tilde{v} \times d^n, \mu \times \delta_n \rangle = (i_* \alpha \times \varepsilon) \circ (i_* \beta \times \delta_n),$$

and it is clear from the special case that this equals $\sum j_p$, since the intersections of A and B on the one hand and $A \times 0$ and $B \times D^n$ on the other hand correspond and have the same indices j_p. (The fact that P and $B \times D^n$ have a boundary makes no difference, that is, in the special case we could have allowed M and B to have a boundary, with $\partial B \subset \partial M$.)

6. REMARK

All the results are easily generalized to the case where N, M (and A, B) have boundaries, with $\partial N \subset \partial M$ and $\partial A, \partial B \subset \partial M$, but $A \cap B \subset M - \partial M$. To construct the Thom class of N we double M, that is, identify M with another copy of M along ∂M, automatically doubling N; we then restrict the Thom class from the doubled manifolds to M (i.e., to (M, M - N)). It is easy to carry out all the necessary steps in this generalized setting; we may assume (a) that ∂N is locally flat in ∂M (if necessary, multiply by D^{n-1}) and (b) that N and ∂M are simultaneously locally flat (in the obvious sense) at each point of ∂N ("expand" M by attaching $I \times \partial M$ to it; in other words, identify ∂M with $0 \times \partial M$; this "expands" N at the same time, and makes the configuration of N and ∂M reasonable at the points of ∂N; the homology, absolute and relative, does not change).

For theorem A we have a commutative diagram

$$\begin{array}{ccccccc} \cdots \to & H^*(M, M - N) & \to & H^*(\partial M, \partial M - \partial N) & \xrightarrow{d^*} & H^*(M, (M - N) \cup \partial M) & \to \cdots \\ & \uparrow u & & \uparrow u & & \uparrow u & \\ \cdots \to & H^*(N) & \to & H^*(\partial N) & \to & H^*(N, \partial N) & \to \cdots, \end{array}$$

where d^* is the usual map

$$H^*(\partial M, \partial M - \partial N) \approx H^*((M - N) \cup \partial M, M - N) \to H^*((M - N) \cup \partial M)$$
$$\to H^*(M, (M - N) \cup \partial M);$$

the last vertical map is a "relative" Thom isomorphism.

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