A CLASS OF RELATION TYPES ISOMORPHIC TO THE ORDINALS

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In this paper, we prove that an ordering relation is scattered and homogeneous if and only if for some ordinal ϕ it is isomorphic to the antilexicographically ordered set of all ϕ-termed sequences of integers that are almost always zero. The algebra of all homogeneous scattered types, under ordinal multiplication, turns out to be isomorphic to the ordinals under ordinal addition.

We shall use the notation of [3] with one exception. For the *ordered sum of the relations* G(x) *over* R, we write $\sum_{x,R} G(x)$ rather than $\sum_{R} G(x)$; analogously, for the *ordinal sum of the types* $\gamma(x)$ *over* R, we write $\sum_{x,R} \gamma(x)$ rather than $\sum_{R} \gamma(x)$. An ordering relation R is (one point) *homogeneous*, if for any x, y ϵ F(R) there exists an automorphism f of R with f(x) = y. Analogously, an order type α is homogeneous if $\alpha = \beta + 1 + \gamma = \beta' + 1 + \gamma'$ implies $\beta = \beta'$ and $\gamma = \gamma'$. We identify the ordinal ϕ with the set of all ordinals less than ϕ. If ϕ has a predecessor, we call the predecessor $\phi - 1$. The symbol $\mathfrak{Z}^\phi$ stands for the set of all functions on ϕ to the set of integers. If $\phi < \rho$, $M \in \mathfrak{Z}^\phi$, $N \in \mathfrak{Z}^\rho$, and $M_\iota = N_\iota$ for every $\iota < \phi$, then we shall refer to N as an *extension of* M. Let ϕ be an ordinal, and let $N \in \mathfrak{Z}^\phi$; then Z^ϕ_N will denote the relation whose field consists of all elements $M \in \mathfrak{Z}^\phi$ such that $M_\iota = N_\iota$ for almost all (all but finitely many) $\iota < \phi$; the elements of $F(Z^\phi_N)$ are ordered antilexicographically. The order type of Z^ϕ_N is obviously the same for any choice of the sequence N. If for N we choose the function on ϕ with range $\{0\}$, we write Z^ϕ_0. Hence, if we use the notation of [2, Chapter VI, Section 3], then

$$\tau(Z^\phi_N) = (\omega^* + 1 + \omega)^\phi_0 .$$

Note that if ϕ is finite, then

$$(\omega^* + 1 + \omega)^\phi_0 = (\omega^* + \omega)^\phi ,$$

and that

$$(\omega^* + 1 + \omega)^0_0 = 1 .$$

Moreover, for any ordinals ϕ and θ,

$$(\omega^* + 1 + \omega)^\phi_0 \cdot (\omega^* + 1 + \omega)^\theta_0 = (\omega^* + 1 + \omega)^{\phi+\theta}_0 ;$$

(see [2, p. 160, (5)]). If there exists a function mapping the ordering relation R isomorphically onto a subrelation of the ordering relation S, we write $R \mathscr{L} S$; if there is no such isomorphism, we write $R \overline{\mathscr{L}} S$. If $\alpha = \tau(R)$ and $\beta = \tau(S)$, we write $\alpha \mathscr{L} \beta$ or

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$\alpha \mathscr{L} \beta$. For any ordinal ϕ, we define R_ϕ as the relation with $F(R_\phi) = \phi$ and the expected ordering, that is, $\iota R_\phi \theta$ if and only if $\iota \leq \theta < \phi$. Put

$$\phi^{(*)} = \{\zeta \mid \zeta = \iota^* \text{ for some } \iota < \phi\}.$$

We define $R_{\phi(*)}$ as the relation with the properties

$$F(R_{\phi(*)}) = \phi^{(*)}, \text{ and } \zeta R_{\phi(*)} \chi \text{ if and only if } \chi^* \leq \zeta^* < \phi.$$

LEMMA 1. *For every ordinal* ϕ, $(\omega^* + 1 + \omega)_0^\phi$ *is homogeneous and scattered.*

Proof. The relation Z_0^ϕ is obviously homogeneous, for, given the points

$$N = (N_0, \cdots, N_\iota, \cdots)_{\iota < \phi}$$

and

$$N' = (N'_0, \cdots, N'_\iota, \cdots)_{\iota < \phi},$$

we can map Z_0^ϕ isomorphically onto itself by the function f defined on $F(Z_0^\phi)$ by

$$f((M_0, \cdots, M_\iota, \cdots)_{\iota<\phi}) = (M_0 + (N'_0 - N_0), \cdots, M_\iota + (N'_\iota - N_\iota), \cdots)_{\iota<\phi}.$$

(For an alternate proof, see Lemma 2(i), (ii) below.)

If $(\omega^* + 1 + \omega)_0^\phi$ is scattered, then

$$(\omega^* + 1 + \omega)_0^{\phi+1} = (\omega^* + 1 + \omega)_0^\phi \cdot (\omega^* + \omega)$$

is also scattered, since, as is well known, scattered types are closed under ordinal multiplication. Now assume that ϕ is a limit ordinal and that $(\omega^* + 1 + \omega)_0^\iota$ is scattered for every $\iota < \phi$. If $(\omega^* + 1 + \omega)_0^\phi$ were not scattered, there would exist points N and N' in $F(Z_0^\phi)$ with the property that $Z_0^\phi \langle [N, N'] \rangle$ contains a dense subrelation. Let δ be the largest ordinal such that $N_\delta \neq 0$ or $N'_\delta \neq 0$. Then the relation $Z_0^\phi \langle [N, N'] \rangle$ is isomorphic to a subinterval of $Z_0^{\delta+1}$, which by hypothesis is scattered. Hence $(\omega^* + 1 + \omega)_0^\phi$ is scattered.

LEMMA 2.

(i) *If* $\alpha = 1 + \alpha'$ *and* $\alpha \mathscr{F} (\omega^* + 1 + \omega)_0^\phi$, *then*

$$\alpha' = \sum_{\iota, R_\phi} [(\omega^* + 1 + \omega)_0^\iota \cdot \omega].$$

(ii) *If* $\alpha = \alpha' + 1$ *and* $\alpha \mathscr{I} (\omega^* + 1 + \omega)_0^\phi$, *then*

$$\alpha' = \sum_{\iota, R_{\phi(*)}} [(\omega^* + 1 + \omega)_0^{\iota^*} \cdot \omega^*].$$

(iii) *Let* $M \in \mathfrak{Z}^{\phi}$, *let* $N Z_M^{\phi} N'$, *and let* δ *be the largest ordinal such that* $N_\delta \neq N'_\delta$. *Then for some finite type* q,

$$\tau(Z_M^{\phi}\langle(N, N')\rangle) = \sum_{\iota, R_\delta} [(\omega^* + 1 + \omega)_0^{\iota} \cdot \omega] + (\omega^* + 1 + \omega)_0^{\delta} \cdot q + \sum_{\iota, R_\delta(*)} [(\omega^* + 1 + \omega)_0^{\iota^*} \cdot \omega^*].$$

Proof. For (i) and (ii), see [7, p. 16, Corollary 1, and p. 17, Corollary 2]; note that

$$\left(\sum_{\iota, R_\delta} [(\omega^* + 1 + \omega)_0^{\iota} \cdot \omega]\right)^* = \sum_{\iota, R_{\delta(*)}} [(\omega^* + 1 + \omega)_0^{\iota^*} \cdot \omega^*].$$

In (iii), we may assume without loss of generality that $N_\iota = M_\iota = 0$ for all $\iota < \phi$. Then δ is the largest ordinal such that $N'_\delta \neq 0$. The interval (N, N') consists of

(a) all sequences $X \in F(Z_0^{\phi})$ such that $X \neq N$, such that $X_\iota = 0$ for $\delta \leq \iota < \phi$, and such that $N Z_0^{\phi} X$,

(b) all sequences $X \in F(Z_0^{\phi})$ such that $X_\delta = m$, where $0 < m < N'_\delta$, and $X_\iota = 0$ for $\delta < \iota < \phi$,

(c) all sequences $X \in F(Z_0^{\phi})$ with $X \neq N'$, $X_\delta = N'_\delta$, $X_\iota = 0$ for $\delta < \iota < \phi$, and $X Z_0^{\phi} N'$.

Thus we can write

$$\tau(Z_0^{\phi}\langle(N, N')\rangle) = \alpha + \beta + \gamma,$$

where α, β, and γ are the types determined by the intervals described in (a), (b), and (c), respectively. From (a) and (i) we get

$$\alpha = \sum_{\iota, R_\delta} [(\omega^* + 1 + \omega)_0^{\iota} \cdot \omega];$$

from (b) we obtain

$$\beta = (\omega^* + 1 + \omega)_0^{\delta} \cdot q, \qquad \text{where } q = N'_\delta - 1;$$

from (c) and (ii) we get

$$\gamma = \sum_{\iota, R_{\delta(*)}} [(\omega^* + 1 + \omega)_0^{\iota^*} \cdot \omega^*].$$

Thus (iii) holds.

We need the following result:

LEMMA 3. *If* α *is a scattered order type, then* $(\alpha + \alpha)\overline{\mathscr{L}}\alpha$.

Proof. See [1, p. 519, Lemma 1.4].

LEMMA 4. *If* $\gamma \cdot \delta$ *is homogeneous and scattered and* $\delta \neq 1$, *then* $\delta = (\omega^* + \omega) \cdot \delta_1$ *for some* δ_1.

Proof. By [3, p. 52, Corollary 3.2], there exists a relation T such that

(1)
$$\delta = \sum_{i,T} \zeta_i ,$$

where each ζ_i is an elementary type, and if i, i' $\in$ F(T) and i immediately precedes i' with respect to T, then $\zeta_i + \zeta_{i'}$ is not an elementary type.

If $\zeta_i = \omega^* + \omega$ for every $i \in F(T)$, then $\delta = (\omega^* + \omega) \cdot \rho$, where $\rho = \tau(T)$, and the Lemma holds. We shall assume that

(2)
$$\zeta_i \neq \omega^* + \omega \quad \text{for some } i \in F(T),$$

and show that (2) leads to a contradiction.

First, consider the case where T is a one-element relation. Then

$$\delta = n > 1, \quad \text{or } \delta = \omega, \quad \text{or } \delta = \omega^* .$$

We show that the first case is impossible; a quite similar argument holds for the other two. If γ is finite, then $\gamma \cdot \delta$ is finite and hence not homogeneous. Hence γ is infinite, and so

(3)
$$\gamma = \gamma_1 + 1 + \gamma_2 \quad \text{for some } \gamma_1 \neq 0 \text{ and } \gamma_2 \neq 0 .$$

Further,

(4)
$$\gamma \cdot \delta = \gamma_1 + 1 + \gamma_2 + \gamma_1 + 1 + \gamma_2 + \gamma \cdot m \quad \text{for some finite type } m .$$

Since $\gamma \cdot \delta$ is homogeneous, it follows from (4) that

$$\gamma_1 = \gamma_1 + 1 + \gamma_2 + \gamma_1 ,$$

and hence

(5)
$$(\gamma_1 + \gamma_1) \mathscr{L} \gamma_1 .$$

By (5) and Lemma 3, γ_1 is not scattered; hence $\gamma \cdot \delta$ is not scattered. Thus (2) leads to a contradiction.

Now assume that F(T) contains more than one element. We need to consider two cases:

(a) $\zeta_i = n$ for some $i \in F(T)$;

(b) $\zeta_i = \omega$ or $\zeta_i = \omega^*$ for some $i \in F(T)$.

Case (a) must be divided into two subcases:

(a_1) either iTi' for every i' $\in$ F(T) or i'Ti for every i' $\in$ F(T);

(a_2) for some $i_1, i_2 \in F(T)$, $i_1 T i$ and $i T i_2$.

Assume first that (a_1) holds and that iTi' for every $i' \in F(T)$. Then

$$\delta = n + \mu, \qquad \text{for some } \mu \neq 0.$$

If i has a successor, say i_1, with respect to T, then $1 \overline{\mathscr{I}} \zeta_{i_1}$, for if we had $1 \mathscr{I} \zeta_1$, then $\zeta_i + \zeta_{i_1}$ would be an elementary type, in contradiction to (1). Hence

(6) $$\delta = n + \mu, \qquad \text{where } 1 \overline{\mathscr{I}} \mu \text{ and } \mu \neq 0.$$

If γ were finite, we would have the relation

$$\gamma \cdot \delta = m + \gamma \cdot \mu, \qquad \text{where m is finite and } m \neq 0,$$

but in this case $\gamma \cdot \delta$ could not be homogeneous. Hence γ is infinite, and so (3) holds. From (6) and (3) one obtains

$$\gamma \cdot \delta = \gamma_1 + 1 + \gamma_2 + \gamma \cdot (n - 1) + \gamma \cdot (\mu_1 + 1 + \mu_2), \qquad \text{with } \mu_1 + 1 + \mu_2 = \mu.$$

From the above equation and the hypothesis we deduce that

$$\gamma_1 = \gamma \cdot n + \gamma \cdot \mu_1 + \gamma_1,$$

and hence (5).

Now assume (a_2). By arguing as we did to obtain (6), we see that

(7) $\delta = \mu + n + \nu$, where μ, ν are nonzero types and $1 \overline{\mathscr{F}} \mu$ and $1 \overline{\mathscr{I}} \nu$.

Since ν is scattered and infinite, (7) implies that

$$\delta = \mu + n + \nu_1 + 2 + \nu_2, \qquad \text{where } \nu_1 \neq 0 \text{ and } 1 \overline{\mathscr{I}} \nu_1.$$

Hence

(8) $$\gamma \cdot \delta = \gamma \cdot \mu + \gamma \cdot n + \gamma \cdot \nu_1 + \gamma + \gamma + \gamma \cdot \nu_2.$$

Now γ is infinite, for if γ were finite we could obtain

$$\gamma \cdot \mu = \gamma \cdot (\mu + n + \nu_1) + \gamma$$

from (8). Since $1 \overline{\mathscr{F}} \mu$, the above identity is absurd. Now, using (8), (3), and the hypothesis, we obtain

(9) $$\gamma \cdot \mu + \gamma_1 = \gamma \cdot (\mu + n + \nu_1) + (\gamma + \gamma_1).$$

For convenience put

(10) $$\rho = \mu + n + \nu_1.$$

From (9), (10), and [3, p. 53, Lemma 3.4], we get either

(11) $\gamma \cdot \mu = \gamma \cdot \rho + \phi$ and $\gamma + \gamma_1 = \phi + \gamma_1$ for some ϕ,

or

(12) $\gamma \cdot \rho = \gamma \cdot \mu + \phi$ and $\gamma_1 = \phi + \gamma + \gamma_1$ for some ϕ.

If (11) holds and $\phi = 0$, we obtain (5) from the second identity of (11). In the case $\phi \neq 0$, we apply [3, p. 53, Lemma 3.5] to the first identity of (11) and obtain either

(13) $\gamma \cdot \rho = \gamma \cdot \mu_1$ and $\phi = \gamma \cdot \mu_2$, where $\mu_1 + \mu_2 = \mu$,

or

(14) $\gamma \cdot \rho = \gamma \cdot \mu_1 + \gamma^{(1)}$ and $\phi = \gamma^{(2)} + \gamma \cdot \mu_2$,
for some types $\gamma^{(1)}$, $\gamma^{(2)}$, μ_1, μ_2, with $\mu_1 + 1 + \mu_2 = \mu$.

From both (13) and (14) we get

$$\gamma \cdot \mu_2 \mathscr{L} \phi,$$

while from the second identity of (11) we obtain

$$\phi \mathscr{L} (\gamma + \gamma_1);$$

therefore

$$\gamma \cdot \mu_2 \mathscr{L} (\gamma + \gamma_1).$$

Now, using (7), we find that μ_2 must be infinite in (13) and in (14). Hence

$$(\gamma + \gamma_1) \cdot 2 \mathscr{L} (\gamma + \gamma_1),$$

and $(\gamma + \gamma_1)$ is not scattered, nor is γ. If (12) holds, then from the second identity of (12) we obtain (5).

We have arrived at (b). It suffices to assume that $\zeta_i = \omega$ for some $i \in F(T)$. Again we need to consider two cases:

(b_1) $\zeta_i = \omega$ and iTi' for every $i' \in F(T)$;

(b_2) $\zeta_i = \omega$ and $i'Ti$ for some $i' \in F(T)$.

In the case (b_1), we have

(15) $\delta = \omega + \nu$ for some ν.

From (15), we readily obtain (3) and then (5). In the case (b_2), we get

(16) $\delta = \mu + \omega + \nu$, for some types μ, ν, with $\mu \neq 0$ and $1 \overline{\mathscr{F}} \mu$,

and hence

(17) $\gamma \cdot \delta = \gamma \cdot \mu + \gamma \cdot \omega + \gamma \cdot \nu$.

If γ is finite, then from (17) and the hypothesis we get

$$\gamma \cdot \mu = \gamma \cdot \mu + \gamma . \tag{18}$$

From (18) and the fact that every scattered type is a left cancelling type (see [3, p. 59, Theorem 3.13]), we obtain

$$\mu = \mu + 1,$$

in contradiction to (16). Thus (3) holds. From (3) and (16) it follows that

$$\gamma \cdot \mu + \gamma_1 = \gamma \cdot \mu + (\gamma + \gamma_1) \quad \text{for some } \gamma_1 \neq 0 . \tag{19}$$

Now by applying [3, p. 53, Lemmas 3.4 and 3.5] to (19), we obtain a contradiction.

LEMMA 5. *Let α be an order type and let ρ be a limit ordinal. If for each $\iota < \rho$ there exists a type γ_ι such that $\alpha = (\omega^* + 1 + \omega)_0^\iota \cdot \gamma_\iota$, then*

$$\alpha = (\omega^* + 1 + \omega)_0^\rho \cdot \gamma_\rho$$

for some type γ_ρ.

Proof. Let R be a relation of type α, and let $\iota < \rho$. By hypothesis, there exist a function f_ι, a relation S_ι, and for each $s \in F(S_\iota)$ a relation $Z^\iota_{\chi(s)}$, where $\chi(s) \in \mathfrak{Z}^\iota$, such that

$$R \underset{f_\iota}{\cong} \sum_{s, S_\iota} Z^\iota_{\chi(s)} . \tag{1}$$

Corresponding to each ordinal ι there may obviously exist various choices of f_ι, S_ι, and the relations $Z^\iota_{\chi(s)}$. We define a partial ordering relation $\lessdot$ on all isomorphism functions satisfying (1) for some $\iota < \rho$ as follows:

(2) $f_\phi \lessdot g_\mu$ provided either $\phi = \mu$ and $f_\phi = g_\mu$; or else $\phi < \mu < \rho$, and for every $r \in F(R)$, if $f_\phi(r) = (M, s_\phi)$ and $g_\mu(r) = (M', s'_\mu)$, with $M \in \mathfrak{Z}^\phi$ and $M' \in \mathfrak{Z}^\mu$, then M' is an extension of M.

Using a familiar form of the axiom of choice, we construct a maximal simply ordered subrelation of $\lessdot$. Denote by Φ the field of this relation. For each $\iota < \rho$ there obviously exists at most one function in Φ satisfying (1). If such a function exists, we denote it by f_ι. We now show that

(3) if $\phi < \mu < \rho$ and $f_\mu \in \Phi$, then there exists an $f_\phi \in \Phi$.

Assume that (3) is false; we can then define an appropriate function f_ϕ. Let $r \in F(R)$, and suppose that

$$f_\mu(r) = (N, s_\mu), \quad \text{where } N \in \mathfrak{Z}^\mu .$$

Define

$$f_\phi(r) = ((N_0, \cdots, N_\iota, \cdots,)_{\iota < \phi}, s_\phi),$$

where

$$s_\phi = ((N_\phi, \cdots, N_\iota, \cdots)_{\phi \leq \iota < \mu}, s_\mu) .$$

We define a relation S_ϕ whose field is the set of all second terms of the ordered pairs $f_\phi(r)$, where $r \in F(R)$. Suppose $r' \in F(R)$ and

$$f_\mu(r') = ((N_0', \cdots, N_\iota', \cdots)_{\iota < \mu}, s_\mu') ,$$

and hence

$$f_\phi(r') = ((N_0', \cdots, N_\iota', \cdots)_{\iota < \phi}, s_\phi') ,$$

where

$$s_\phi' = ((N_\phi', \cdots, N_\iota', \cdots)_{\phi \leq \iota < \mu}, s_\mu') .$$

If $s_\mu \neq s_\mu'$ and $s_\mu S_\mu s_\mu'$ in (1), we put

(4) $$s_\phi S_\phi s_\phi' .$$

If $s_\phi = s_\phi'$, then the sequences

(5) $$(N_\phi, \cdots, N_\iota, \cdots)_{\phi \leq \iota < \mu} \quad \text{and} \quad (N_\phi', \cdots, N_\iota', \cdots)_{\phi \leq \iota < \mu}$$

can differ at only a finite number of places. In this case, (4) holds if and only if the first sequence of (5) precedes the second with respect to antilexicographic ordering.

Now suppose that the sequence

$$M = (M_0, \cdots, M_\iota, \cdots)_{\iota < \phi}$$

differs from the first sequence of (5) at only a finite number of places. Since there exists an $r^* \in F(R)$ with

$$f_\mu(r^*) = ((M_0, \cdots, M_\iota, \cdots)_{\iota < \mu}, s_\mu) ,$$

where

$$M_\iota = N_\iota \quad \text{for } \phi \leq \iota < \mu ,$$

it follows that

$$f_\phi(r^*) = ((M_0, \cdots, M_\iota, \cdots)_{\iota < \phi}, s_\phi) .$$

One easily concludes that $f_\phi \preccurlyeq f_\mu$ and that (3) holds.

We now show that

(6) $$\text{if } \mu < \rho \text{ and } f_\mu \in \Phi , \text{ then } f_{\mu+1} \in \Phi .$$

By (6),

$$R \underset{f_\mu}{\cong} \sum_{s, S_\mu} Z^\mu_{\chi(s)}, \qquad \text{where } \chi(s) \in \mathfrak{Z}^\mu \text{ for every } s \in F(S_\mu),$$

and hence

$$\alpha = (\omega^* + 1 + \omega)_0^\mu \cdot \delta, \text{ where } \tau(\delta) = S_\mu .$$

By the hypothesis of the lemma,

$$\alpha = (\omega^* + 1 + \omega)_0^\mu \cdot \gamma_\mu = (\omega^* + 1 + \omega)_0^{\mu+1} \cdot \gamma_{\mu+1} .$$

Using the fact that $(\omega^* + 1 + \omega)_0^\mu$ is a left cancelling type, we find that

$$S_\mu \underset{g}{\cong} Z_0^1 \cdot T \qquad \text{for some function } g \text{ and some relation } T.$$

Now suppose that $r \in F(R)$, that

$$f_\mu(r) = (N, s_\mu), \qquad \text{where } N \in \mathfrak{Z}^\mu \text{ and } s_\mu \in F(S_\mu),$$

and that

$$g(s_\mu) = (m, t).$$

We put

$$f_{\mu+1}(r) = (N^*, t), \qquad \text{where } N^* \in \mathfrak{Z}^{\mu+1},$$

$$N^*_\iota = N_\iota \quad \text{for } 0 \leq \iota < \mu \quad \text{and } N^*(\mu) = m .$$

Thus (6) holds.

Now we need the following:

(7) If μ is a limit ordinal and if $f_\iota \in \Phi$ for each $\iota < \mu$, then there exists an $f_\mu \in \Phi$.

Assume that (7) fails. Then by (3) and (7),

$$\Phi = \{f_\iota \mid \iota < \kappa\} \qquad \text{for some limit ordinal } \kappa .$$

With each $r \in F(R)$ we correlate two elements of $\mathfrak{Z}^\kappa$, namely $N^{(r)}$ and $S^{(r)}$; $N^{(r)}$ is the (unique) extension of all first terms of the ordered pairs $f_\iota(r)$ for $\iota < \kappa$; and, for each $\iota < \kappa$, $S^{(r)}_\iota$ is the second term of the ordered pair $f_\iota(r)$.

Suppose that

(8) $$rRr', \qquad \phi < \mu < \kappa, \qquad \text{and } S^{(r)}_\phi = S^{(r')}_\phi .$$

Let δ be the largest ordinal such that

$$\delta < \phi \qquad \text{and} \qquad N^{(r)}_\delta \neq N^{(r')}_\delta .$$

Using (8), we obtain

$$\tau(R\langle(r, r')\rangle)\mathscr{L}(\omega^* + 1 + \omega)_0^{\delta+1},$$

and hence

(9) $$\tau(R\langle(r, r')\rangle)\mathscr{L}(\omega^* + 1 + \omega)_0^{\phi}.$$

Assume that

$$s_\mu \neq s'_\mu .$$

Then from Lemma 2(i), (ii) it follows that

(10) $$\left(\sum_{\iota, R_\mu(*)} [(\omega^* + 1 + \omega)_0^{\iota^*} \cdot \omega^*] + \sum_{\iota, R_\mu} [(\omega^* + 1 + \omega)_0^{\iota} \cdot \omega]\right) \mathscr{L}\tau(R\langle(r, r')\rangle).$$

Using (8), (9), and (10), we get

$$((\omega^* + 1 + \omega)_0^{\phi} \cdot 2)\mathscr{L}(\omega^* + 1 + \omega)_0^{\phi},$$

which is obviously false. Now we conclude that

(11) if $r, r' \in F(R)$ and $S_\phi^{(r)} = S_\phi^{(r')}$ for some $\phi < \kappa$, then $S_\mu^{(r)} = S_\mu^{(r')}$ for all μ such that $\phi < \mu < \kappa$.

By an argument that is quite similar to the derivation of (11), but in which one uses (11) and Lemma 2(iii) rather than Lemma 2(i), (ii), one obtains the following:

(12) If $r, r' \in F(R)$ and if $S_\phi^{(r)} = S_\phi^{(r')}$ for some $\phi < \kappa$, then $N_\mu^{(r)} = N_\mu^{(r')}$ for all μ such that $\phi < \mu < \kappa$.

Now put

(13) $r \sim r'$ if $r, r' \in F(R)$ and $S_\iota^{(r)} = S_\iota^{(r')}$ for some $\iota < \kappa$.

It follows from (11) that $\sim$ is an equivalence relation over $F(R)$. Note that each equivalence class $r/\sim$ is an interval of R. Let S_κ be a relation with

$$F(S_\kappa) = \{r/\sim \mid r \in F(R)\},$$

and let

$$(r/\sim, r'/\sim) \in S_\kappa \text{ if } r/\sim = r'/\sim \text{ or if } r/\sim \text{ precedes } r'/\sim \text{ in } R.$$

We can now define a function f_κ on $F(R)$ as follows:

(14) $$f_\kappa(r) = (N^{(r)}, r/\sim).$$

If $r \sim r'$, then from (13) and (12) we find that $N_\iota^{(r)} = N_\iota^{(r')}$ for almost all ι. On the other hand, suppose that

$$r \in F(R),\ N \in \mathfrak{Z}^\kappa,\ N \neq N^{(r)},\ \text{and } N_\iota = N_\iota^{(r)} \text{ for almost all } \iota.$$

Let δ be the largest ordinal such that $N_\delta \neq N_\delta^{(r)}$, and suppose that

$$f_{\delta+1}(r) = ((N_0^{(r)}, \cdots, N_\iota^{(r)}, \cdots)_{\iota<\delta+1}, s_\delta).$$

By (1), there exists an $r^* \in F(R)$ such that

$$f_{\delta+1}(r^*) = ((N_0, \cdots, N_\iota, \cdots)_{\iota<\delta+1}, s_\delta).$$

It follows from (13) and (11) that

$$f_\kappa(r^*) = (N, r/\sim).$$

For each $s \in F(S_\kappa)$ choose an appropriate sequence $\chi(s) \in \mathfrak{Z}^\kappa$. Using (14), we obtain

$$R \underset{f_\kappa}{\cong} \sum_{s, S_\kappa} Z^\kappa_{\chi(s)};$$

from the isomorphism above we conclude (7). The lemma follows from (6) and (7).

THEOREM. *Let α be any order type. Then α is homogeneous and scattered if and only if*

$$\alpha = (\omega^* + 1 + \omega)_0^\rho \quad \textit{for some ordinal } \rho.$$

For each such α there is only one ρ.

Proof. Suppose α is homogeneous and scattered. Let μ be the smallest ordinal with the property

(1) $$\alpha \neq (\omega^* + 1 + \omega)_0^\mu \cdot \theta \quad \text{for every type } \theta.$$

Obviously $\mu \neq 0$; by Lemma 5,

(2) $$\alpha = (\omega^* + 1 + \omega)_0^{\mu-1} \cdot \phi \quad \text{for some } \phi.$$

If $\phi = 1$, our conclusion is at hand. Assume that $\phi \neq 1$; then from (2) and Lemma 4 we obtain

$$\phi = (\omega^* + \omega) \cdot \phi' \quad \text{for some } \phi',$$

and hence

$$\alpha = (\omega^* + 1 + \omega)_0^{\mu-1} \cdot (\omega^* + \omega) \cdot \phi' = (\omega^* + 1 + \omega)_0^\mu \cdot \phi',$$

in contradiction to (1). The uniqueness of the exponent is obvious. Reference to Lemma 1 completes the proof.

The theorem above was stated in the abstract [4]. A related result was proved in [7]; translated into the terminology of order types, the latter result is the following: an order type $\alpha \neq 0$ is *perfectly symmetric* if whenever $\alpha = \beta + \gamma$, $\beta \neq 0$, and $\gamma \neq 0$, then $\beta^* = \gamma$. It is shown that a type α is perfectly symmetric if and only if $\alpha = (\omega^* + 1 + \omega)_0^\rho$ for some ordinal ρ. (There is a slight flaw in this formulation, since the type 2 is an exception.) One might suppose that one could prove the following equivalence without using either the result of [7] or the theorem above:

α is homogeneous and scattered if and only if

$\alpha \neq 2$ and α is perfectly symmetric.

However, neither the author nor the referee was able to do so.

Let HS denote the class of homogeneous scattered types.

COROLLARY 1. *The algebras* (Ordinals, +) *and* (HS, $\cdot$) *are isomorphic.*

Proof. From the theorem.

Now let

$$\mathrm{GS} = \{\alpha \mid \text{for some group } \mathfrak{G}, \text{ there exists a scattered linear ordering R of } \mathfrak{G} \text{ and } \tau(\mathrm{R}) = \alpha\}.$$

COROLLARY 2. GS = HS.

Proof. Trivially, $(\omega^* + 1 + \omega)_0^0$ is the ordering of the one-element group, while if $\rho > 0$, then $(\omega^* + 1 + \omega)_0^\rho$ is the type of an ordering of the free Abelian group with $\mathfrak{m}$ generators, where $\mathfrak{m}$ is the cardinality of ρ. Hence HS $\subseteq$ GS. Since every linear ordering of a group must be homogeneous, GS $\subseteq$ HS.

Our final corollary is a metamathematical remark. The reader is referred to the Introduction of [6] and to [5] for the relevant definitions.

COROLLARY 3. (i) *Let* R *and* S *be homogeneous scattered relations with* $\tau(\mathrm{R}) \neq 1$ *and* $\tau(\mathrm{S}) \neq 1$. *Then* R *and* S *are elementarily equivalent in the arithmetical theory of one binary relation.*

(ii) *The class of homogeneous scattered relations is not elementarily closed in the arithmetical theory of one binary relation.*

Proof. The statement (i) is an almost immediate consequence of [5, p. 228, Theorem 2.12]. Now suppose that $\tau(\mathrm{R}) = \omega^* + \omega$ and that $\tau(\mathrm{T}) = (\omega^* + \omega) \cdot 2$. It follows from the theorem just quoted that R and T are elementarily equivalent; on the other hand, T is not homogeneous.

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