ENDOMORPHISM RING OF AN INDUCED MODULE

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1. INTRODUCTION

Let G be a finite group with a normal subgroup H. Let K be a field, and let L be a left KH-module. In this paper, we investigate the structure of $\mathrm{Hom}_{KG}(L^G, L^G)$.

Let S be the subgroup of G/H = B such that

$$S = \{b \in B \mid \bar{b} \otimes L \text{ is KH-isomorphic to } L\}.$$

If $\mathrm{Hom}_{KG}(\bar{b} \otimes L, L) = 0$ for all $b \in B$, $b \notin S$, then $\mathrm{Hom}_{KG}(L^G, L^G)$ is the crossed product of $\mathrm{Hom}_{KH}(L, L)$ and S (described in Theorem 1).

We apply this to the case where K is algebraically closed and of characteristic $p > 0$, and L is an indecomposable left KH-module. We prove that if

$$\mathrm{Hom}_{KH}(\bar{b} \otimes L, L) = 0$$

for all $b \in B$, $b \notin S$, then L^G is indecomposable if and only if S is a p-group or $S = \{1\}$.

J. A. Green in [3, Theorem 8] proves the following: *If* K *is an algebraically closed field of characteristic* $p > 0$, G *is a* p*-group with subgroup* H, *and* L *is any indecomposable left* KH*-module, then* L^G *is indecomposable.* For the proof, he reduces the theorem to the case where H is maximal, and thus normal, in G, and G/H = S. He then uses a construction similar to the one presented here, for the case where S is a cyclic group of order p.

The basic definitions and notations used in this paper may be found in Curtis and Reiner [2]. All modules considered will be assumed to be unital and finite-dimensional vector spaces over K.

2. DECOMPOSITION OF ELEMENTS OF $\mathrm{Hom}_{KG}(L^G, L^G)$

Let G be a finite group with a normal subgroup H, and let G be the extension of H by $B = G/H = \{b_1 = 1, b_2, \cdots, b_n\}$. Then to each element b_i of B there corresponds a left coset $\bar{b}_i H$ of H in G, and $G = H \cup \bar{b}_2 H \cup \cdots \cup \bar{b}_n H$, where $\bar{b}_1 = 1$.

Multiplication of elements of G is given by

$$\bar{b}h \cdot \bar{b}'h' = \overline{bb'}(b, b')h^{b'}h',$$

where $b, b' \in B$, $h, h' \in H$, $(b, b') \in H$, and $h^b = (\bar{b})^{-1} h\bar{b}$. Furthermore,

$$(1, b) = (b, 1) = 1, \qquad (h^b)^{b'} = (b, b')^{-1} h^{bb'}(b, b'),$$

$$(bb', b'')(b, b')^{b''} = (b, b'b'')(b', b'').$$

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Let K be a field, and let KH denote the group algebra of H over K. Let L be a left KH-module, and let L^G denote the induced module $KG \otimes_{KH} L$. Then $L^G = \Sigma_{i=1}^n \bar{b}_i \otimes L$.

Let $S = \{b \in B \mid \bar{b} \otimes L$ is KH-isomorphic to $L\}$. Then S is a subgroup of B. The isomorphisms of $\bar{b} \otimes L$ onto L are determined by the set of nonsingular linear transformations on L. If D_b is a nonsingular linear transformation on L that determines a KH-isomorphism of $\bar{b} \otimes L$ onto L by $\bar{b} \otimes \ell \to D_b \ell$, then $hD_b = D_b h^b$. The transformation D_b will be considered to be in $\mathrm{Hom}_{KH}(\bar{b} \otimes L, L)$.

Assume that the elements of B have been ordered in such a way that $S = \{b_1 = 1, b_2, \cdots, b_m\}$, where $m \leq n$. Set $b_{ij} = b_i b_j$, where ij is some integer $(1 \leq ij \leq n)$. Since S is a group, it follows that if $1 \leq i, j \leq m$, then $1 \leq ij \leq m$. For each $b_i \in S$, select a nonsingular linear transformation D_i on L such that $hD_i = D_i h^{b_i}$ and $D_1 = 1_L$.

Let $\theta \in \mathrm{Hom}_K(L^G, L^G)$. Then θ can be represented by an $n \times n$ matrix $[\theta_{ij}]$, where

$$\theta_{ij} \in \mathrm{Hom}_K(L, L) \quad \text{and} \quad \theta(\bar{b}_i \otimes \ell) = \sum_{j=1}^{n} \bar{b}_j \otimes \theta_{ji} \ell .$$

If $\sigma \in \mathrm{Hom}_K(L, L)$, then $\bar{\sigma}\theta = [\sigma \theta_{ij}]$.

If $\theta \in \mathrm{Hom}_K(L^G, L^G)$, then $\theta \in \mathrm{Hom}_{KG}(L^G, L^G)$ if and only if $h\theta = \theta h$ for all $h \in H$, and $\bar{b}_k \theta = \theta \bar{b}_k$ $(1 \leq k \leq n)$. (Here h and $\bar{b}_k$ are considered as linear transformations on L^G.)

Because

$$h\theta(\bar{b}_i \otimes \ell) = h\left(\Sigma \bar{b}_j \otimes \theta_{ji} \ell\right) = \Sigma \bar{b}_j \otimes h^{b_j} \theta_{ji} \ell$$

and

$$\theta h(\bar{b}_i \otimes \ell) = \theta(\bar{b}_i \otimes h^{b_i} \ell) = \Sigma \bar{b}_j \otimes \theta_{ji} h^{b_i} \ell ,$$

the relation $h\theta = \theta h$ is equivalent to the condition

$$h^{b_j} \theta_{ji} = \theta_{ji} h^{b_i} \quad (1 \leq i, j \leq n).$$

This condition implies that

$$D_j \theta_{ji} D_i^{-1} = \sigma_{ji} \in \mathrm{Hom}_{KH}(L, L) \quad (1 \leq i, j \leq m)$$

and

$$\theta_{1i} = \sigma_{1i} D_i = \sigma_i D_i \quad (1 \leq i \leq m).$$

Since

$$\bar{b}_k \theta(\bar{b}_i \otimes \ell) = \bar{b}_k \left(\Sigma \bar{b}_j \otimes \theta_{ji} \ell\right) = \Sigma \bar{b}_{kj} \otimes (b_k, b_j) \theta_{ji} \ell$$

and

$$\theta \bar{b}_k(\bar{b}_i \otimes \ell) = \theta(\bar{b}_{ki} \otimes (b_k, b_i)\ell) = \sum \bar{b}_r \otimes \theta_{r,ki}(b_k, b_i)\ell,$$

the relation $\bar{b}_k \theta = \theta \bar{b}_k$ is equivalent to the condition

$$(b_k, b_j)\theta_{ji} = \theta_{kj,ki}(b_k, b_i) \qquad (1 \le k \le n).$$

It follows that $\theta_{k,ki} = \theta_{1i}(b_k, b_i)^{-1}$ $(1 \le k \le n)$.

Now, for $i > m$, let $\theta_{1i} = \zeta_i$. Then $h\zeta_i = \zeta_i h^{b_i}$, and $\bar{b}_i \otimes \ell \to \zeta_i \ell$ defines an element of $\mathrm{Hom}_{KH}(\bar{b}_i \otimes L, L)$. Consider ζ_i to be in $\mathrm{Hom}_{KH}(\bar{b}_i \otimes L, L)$.

Combining these relations, we see that for $\theta \in \mathrm{Hom}_{KG}(L^G, L^G)$,

$$\theta_{k,ki} = \begin{cases} \sigma_i D_i (b_k, b_i)^{-1} & (1 \le i \le m), \\ \zeta_i (b_k, b_i)^{-1} & (i > m), \end{cases}$$

where $\sigma_i \in \mathrm{Hom}_{KH}(L, L)$ and $D_i, \zeta_i \in \mathrm{Hom}_{KH}(\bar{b}_i \otimes L, L)$.

Let V_i be the $n \times n$ matrix with $(b_k, b_i)^{-1}$ in the k, ki-position $(1 \le k \le n)$ and with zeros elsewhere. Then, for $\theta \in \mathrm{Hom}_{KG}(L^G, L^G)$,

$$\theta = \bar{\sigma}_1 + \bar{\sigma}_2 \bar{D}_2 V_2 + \cdots + \bar{\sigma}_m \bar{D}_m V_m + \bar{\zeta}_{m+1} V_{m+1} + \cdots + \bar{\zeta}_n V_n. \tag{1}$$

3. LEMMAS FOR SECTION 4

Lemmas 1 to 3 can be proved by considering the entries in the appropriate matrices. Lemmas 4 to 6 follow immediately from the definitions.

LEMMA 1. $\bar{\theta}_{1i} V_i \in \mathrm{Hom}_{KG}(L^G, L^G)$ $(1 \le i \le n)$.

LEMMA 2. $(\bar{\theta}_{1i} V_i)(\bar{\theta}_{1j} V_j) = \bar{\theta}_{1i} \bar{\theta}_{1j} \overline{(b_i, b_j)^{-1}} V_{ij}$ $(1 \le i, j \le n)$.

Now, define σ^b and $\rho(b, b') \in \mathrm{Hom}_{KH}(L, L)$ by

$$\sigma^b = D_b \sigma D_b^{-1} \qquad (b \in S,\ \sigma \in \mathrm{Hom}_{KH}(L, L)),$$

$$\rho(b, b') = D_b D_{b'} (b, b')^{-1} D_{bb'}^{-1}, \qquad (b, b' \in S).$$

LEMMA 3. *Let* $\sigma \in \mathrm{Hom}_{KH}(L, L)$ *and* $b \in S$. *Then* $(\bar{D}_b V_b)\bar{\sigma} = \bar{\sigma}^b (\bar{D}_b V_b)$, *where* $\bar{\sigma}^b = \overline{\sigma^b}$.

LEMMA 4. $(\bar{D}_b V_b)(\bar{D}_{b'} V_{b'}) = \overline{\rho(b, b')}(\bar{D}_{bb'} V_{bb'})$ $(b, b' \in S)$.

LEMMA 5. $\rho(b', b'')^b \rho(b, b' b'') = \rho(b, b')\rho(bb', b'')$.

LEMMA 6. $(\sigma^b)^{b'} = \rho(b', b)\sigma^{b'b}\rho(b', b)^{-1}$.

4. THE CROSSED PRODUCT OF $\mathrm{Hom}_{KH}(L, L)$ AND S

THEOREM 1. $\mathscr{S} = \left\{ \sum_{b \in S} \bar{\sigma}_b (\bar{D}_b V_b) \mid \sigma_b \in \mathrm{Hom}_{KH}(L, L) \right\}$ *is the crossed product of* $\mathrm{Hom}_{KH}(L, L)$ *and* S *with factor set* ρ *and correspondence* ϕ: $b \in S \to b^*$.

The factor set ρ *is the mapping of the cartesian product* $S \times S$ *into the set of invertible elements in* $\mathrm{Hom}_{KH}(L, L)$ *given by* $\rho(b, b') = D_b D_{b'}(b, b')^{-1} D_{bb'}^{-1}$. *The automorphism* b^* *of* $\mathrm{Hom}_{KH}(L, L)$ *is defined by* $b^*\colon \sigma \to \sigma^b = D_b \sigma D_b^{-1}$.

Proof. If $\mathscr{S}$ is the crossed product as described, then $\mathscr{S}$ must have the following structure (see [4, pp. 81-82]):

$\mathscr{S}$ consists of formal sums $\Sigma \bar{\sigma}_b(\bar{D}_b V_b)$ with $\sigma_b \in \mathrm{Hom}_{KH}(L, L)$, each sum being taken over all $b \in S$. The $\bar{D}_b V_b$ are in one-to-one correspondence with the elements of S, and

(i) $\sum \bar{\sigma}_b(\bar{D}_b V_b) = \sum \bar{\sigma}'_b(\bar{D}_b V_b)$ if and only if $\sigma_b = \sigma'_b$, for every $b \in S$,

(ii) $$\bar{\lambda}\left(\sum \bar{\sigma}_b(\bar{D}_b V_b)\right) + \bar{\lambda}'\left(\sum \bar{\sigma}'_b(\bar{D}_b V_b)\right) = \sum (\overline{\lambda\sigma_b + \lambda'\sigma'_b})(\bar{D}_b V_b)$$
$$(\lambda, \lambda' \in \mathrm{Hom}_{KH}(L, L)),$$

(iii) $$\left(\sum \bar{\sigma}_b(\bar{D}_b V_b)\right) \cdot \left(\sum \bar{\sigma}'_{b'}(\bar{D}_{b'} V_{b'})\right) = \sum_{b,b'} \overline{\sigma_b(\sigma'_{b'})^b \rho(b, b')}(\bar{D}_{bb'} V_{bb'}).$$

The correspondence $b \to \bar{D}_b V_b$ is one-to-one, since for $i \neq j$ the nonzero entries in the first row of $\bar{D}_i V_i$ and $\bar{D}_j V_j$ occur in different columns. In (i), the entry in the 1, i-position on the left is $\sigma_i D_i$; on the right it is $\sigma'_i D_i$. Since D_i^{-1} exists, $\sigma_i = \sigma'_i$. Condition (ii) follows immediately from the properties of $\mathrm{Hom}_K(L^G, L^G)$; (iii) we obtain by applying Lemmas 3 and 4. Lemmas 5 and 6 give the fundamental relationships involving the factor set and the correspondence.

Remark. The structure of $\mathscr{S}$ does not depend upon the choice of the transformations D_b, since any other choice gives a crossed product strictly equivalent to $\mathscr{S}$.

5. CONDITIONS THAT GIVE $\mathrm{Hom}_{KG}(L^G, L^G)$ AS A CROSSED PRODUCT

The following theorem follows immediately from (1) and Theorem 1.

THEOREM 2. $\mathrm{Hom}_{KG}(L^G, L^G)$ *is the crossed product of* $\mathrm{Hom}_{KH}(L, L)$ *and* S *described in Theorem* 1 *if and only if* $\mathrm{Hom}_{KH}(\bar{b} \otimes L, L) = 0$, *for all* $b \in B$, $b \notin S$.

COROLLARY 1. *If* $S = B$, *then* $\mathrm{Hom}_{KG}(L^G, L^G)$ *is the crossed product of* $\mathrm{Hom}_{KH}(L, L)$ *and* B.

This corollary is true since every b in B belongs to S. J. A. Green in [3, Theorem 8] essentially proves this for the case in which $S = B$ is a cyclic group of prime order p.

COROLLARY 2. *Let* K *be algebraically closed, and let* L *be completely reducible. Then* $\mathrm{Hom}_{KG}(L^G, L^G)$ *is the crossed product of* $\mathrm{Hom}_{KH}(L, L)$ *and* S *if and only if, whenever* $b \in B$ *and* $b \notin S$, $\bar{b} \otimes L$ *and* L *are disjoint left* KH-*modules, that is, have no composition factors in common.*

Proof. Under the conditions given, it follows from [2, Section 43] that $\mathrm{Hom}_{KH}(\bar{b} \otimes L, L) = 0$ if and only if $\bar{b} \otimes L$ and L are disjoint left KH-modules.

COROLLARY 3. *Let* K *be algebraically closed, and let* L *be irreducible. Then* $\mathrm{Hom}_{KG}(L^G, L^G)$ *is the crossed product of* K *and* S *with factor set* ρ *and trivial correspondence; in other words, it is the* ρ-*twisted group algebra of* S *over* K.

Proof. By Schur's Lemma, $\mathrm{Hom}_{KH}(L, L) = K \cdot 1_L$ and

$$\mathrm{Hom}_{KH}(\bar{b} \otimes L, L) = 0, \qquad \text{for } b \in B,\ b \notin S.$$

Here $b \to D_b$ is a projective representation of S. with factor set ρ, and b^* is the identity automorphism.

6. STRUCTURE OF $\mathscr{S}$ WHEN L IS INDECOMPOSABLE AND K IS ALGEBRAICALLY CLOSED

In order to simplify notation, replace $\overline{D}_b V_b$ by b and $\overline{\sigma}$ by σ, in the definition of $\mathscr{S}$. Then

$$\mathscr{S} = \left\{ \sum_{b \in S} \sigma_b b \,\middle|\, \sigma_b \in \mathrm{Hom}_{KH}(L, L) \right\},$$

where $b \cdot b' = \rho(b, b')bb'$ and $b\sigma = \sigma^b b$.

$\mathrm{Hom}_{KH}(L, L)$ is a finite-dimensional algebra over the field K. Let N be its radical. We can define a homomorphism of $\mathscr{S}$ by reducing the coefficients modulo N, that is, by the mapping

$$\sum \sigma_b b \to \sum (\sigma_b + N) b = \sum \sigma_b b + \sum N b.$$

Let $\mathscr{T} = \{\Sigma(\sigma_b + N)b\}$. Then $\mathscr{T}$ is the crossed product of $\mathrm{Hom}_{KH}(L, L)/N$ and S with factor set $\rho(b, b') + N$ and correspondence

$$b \to b^{\#}: \sigma + N \to \sigma^b + N.$$

The kernel of this homomorphism is ΣNb. Since $(\Sigma Nb)^j \subseteq \Sigma N^j b$, this is a nilpotent ideal and is thus contained in the radical $N(\mathscr{S})$ of $\mathscr{S}$. The radical of $\mathscr{T}$ is $N(\mathscr{T}) = N(\mathscr{S})/\Sigma Nb$. Thus $\mathscr{S}/N(\mathscr{S}) \cong \mathscr{T}/N(\mathscr{T})$.

Assume L is indecomposable. Then $\mathrm{Hom}_{KH}(L, L)$ is completely primary; that is, $\mathrm{Hom}_{KH}(L, L)/N$ is a division ring (see [1, p. 97]). If we further assume that K is algebraically closed, then the division ring is isomorphic to K, and $\mathscr{T}$ is then isomorphic to the crossed product of K and S with a K-factor set β and trivial correspondence. (The correspondence is trivial, since the automorphisms involved are actually K-automorphisms.) Thus $\mathscr{T}$ is isomorphic to $(KS)_\beta$, the β-twisted group algebra of S over K.

Remark. β can be obtained in the following way: $\rho(b, b') = \alpha(b, b') + n(b, b')$, where $\alpha(b, b') \in K \cdot 1'$ ($1'$ is the identity in $\mathrm{Hom}_{KH}(L, L)$) and $n(b, b') \in N$. α is actually a factor set of S. Define $\beta(b, b') \in K$ by $\alpha(b, b') = \beta(b, b') \cdot 1'$.

LEMMA 7. *Let* K *be algebraically closed, and let* L *be indecomposable. Then* $\mathscr{S}$ *is completely primary if and only if* $(KS)_\beta$ *is completely primary.*

THEOREM 3. *Let* K *be an algebraically closed field of characteristic* $p > 0$. *Then* $(KS)_\beta$ *is completely primary if and only if* S *is a* p-*group or* $S = \{1\}$.

Proof. Let S be a p-group. It is well known [2, p. 189] that if β is the trivial factor set 1, that is, if $\beta(b, b') = 1$ for all $b, b' \in S$, then KS is completely primary. If S is a p-group, this is the only case that can arise, since each β is then equivalent to 1, as can be seen if we consider the multiplier M of S: Let $\{\beta\}$ denote the equivalence class of factor sets of S containing β. It is proved in [2, Section 53] that the order h of $\{\beta\}$ divides the order of S, which is p^e, and that h is not divisible by the characteristic of K, which is p. Thus $h = 1$, and $\{\beta\} = \{1\}$.

If $S = \{1\}$, then $(KS)_\beta$ is K, which is completely primary.

Suppose that $S \neq \{1\}$ is not a p-group. Let $Q \neq \{1\}$ be a q-Sylow subgroup of S, for some $q \neq p$. Define $(KQ)_\beta$ by using the multiplication in $(KS)_\beta$. Then $(KQ)_\beta$ is semisimple (see [4, p. 80]). Since $(KQ)_\beta \neq K \cdot 1$, it contains an idempotent other than $\beta(1, 1)^{-1} \cdot 1$, which is the identity of $(KS)_\beta$. Thus $(KS)_\beta$ contains at least two idempotents and is not completely primary.

L^G is indecomposable if and only if $\mathrm{Hom}_{KG}(L^G, L^G)$ is completely primary. So, combining Lemma 7 and Theorem 3 with the condition that gives $\mathscr{S}$ is $\mathrm{Hom}_{KG}(L^G, L^G)$, we obtain the following result.

THEOREM 4. *Let* K *be an algebraically closed field of characteristic* $p > 0$. *Let* G *be a finite group with a normal subgroup* H. *Let* L *be an indecomposable left* KH-*module. Let*

$$S = \{b \in G/H \mid \bar{b} \otimes L \textit{ is } KH\textit{-isomorphic to } L\}.$$

Assume that $\mathrm{Hom}_{KH}(\bar{b} \otimes L, L) = 0$ *whenever* $b \in G/H$ *and* $b \notin S$. *Then* L^G *is indecomposable if and only if* S *is a* p-*group or* $S = \{1\}$.

COROLLARY. *Let* K *be an algebraically closed field of characteristic* $p > 0$, *and let* L *be an irreducible left* KH-*module. Then* L^G *is indecomposable if and only if* S *is a* p-*group or* $S = \{1\}$.

Remark. Let K be algebraically closed and of characteristic 0, and let L be an irreducible left KH-module. L^G is irreducible if and only if

$$\mathrm{Hom}_{KG}(L^G, L^G) = K \cdot 1'',$$

where $1''$ is the identity map on L^G. However, the case

$$K \cdot 1'' \subseteq \mathscr{S} \subseteq \mathrm{Hom}_{KG}(L^G, L^G) = K \cdot 1''$$

occurs if and only if $S = \{1\}$ and $\mathscr{S} = \mathrm{Hom}_{KG}(L^G, L^G)$. Thus L^G is irreducible if and only if $\mathrm{Hom}_{KH}(\bar{b} \otimes L, L) = 0$ whenever $b \neq 1$ and $b \in G/H$. This is a well-known theorem (see [2, Section 45]).

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