A STABILITY CONDITION FOR THE DIFFERENTIAL EQUATION $y'' + p(x)y = 0$

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In this note we give a simple condition for all solutions of the differential equation

$$y'' + p(x)y = 0 \tag{L}$$

to tend to zero as x tends to infinity.

THEOREM. *If* $p(x) > 0$, $p(x) \in C^3[a, \infty)$, $\lim_{x \to +\infty} p(x) = +\infty$, *and*

$$\int_a^\infty \left| \left(\frac{1}{\sqrt{p(x)}} \right)''' \right| dx < \infty,$$

then $\lim_{x \to +\infty} y(x) = 0$ *for each solution* $y(x)$ *of* (L).

For the literature on the asymptotic behavior of the solutions of (L) under the hypothesis that $p(x) \to +\infty$ as $x \to +\infty$, the reader may consult [1, Section 5.5].

Proof. If $y(x)$ is a solution of (L), then, as may be verified by differentiation,

$$\begin{aligned} w(x) &\equiv \frac{[y'(x)]^2}{\sqrt{p(x)}} - \left(\frac{1}{\sqrt{p(x)}} \right)' y(x) y'(x) + \left[\sqrt{p(x)} + \frac{1}{2} \left(\frac{1}{\sqrt{p(x)}} \right)'' \right] [y(x)]^2 \\ &= w(a) + \frac{1}{2} \int_a^x \left(\frac{1}{\sqrt{p(t)}} \right)''' [y(t)]^2 \, dt. \end{aligned} \tag{1}$$

We shall first establish boundedness of $y(x)$ on the interval I: $a \leq x \leq +\infty$. Since $(1/\sqrt{p(x)})'''$ is absolutely integrable, $(1/\sqrt{p(x)})''$ is bounded on I. Therefore, since $\lim_{x \to +\infty} p(x) = +\infty$,

$$\lim_{x \to +\infty} \left[\sqrt{p(x)} + \frac{1}{2} \left(\frac{1}{\sqrt{p(x)}} \right)'' \right] = +\infty. \tag{2}$$

By (2) and the condition

$$\int_a^\infty \left| \left(\frac{1}{\sqrt{p(x)}} \right)''' \right| dx < +\infty,$$

there exists a number $b \in [a, \infty)$ such that

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(3) $$\left[\sqrt{p(x)} + \left(\frac{1}{\sqrt{p(x)}}\right)''\right] > 2$$

and

(4) $$\int_b^x \left|\left(\frac{1}{\sqrt{p(x)}}\right)'''\right| dx < 1$$

for all $x \geq b$.

Since $\lim_{x \to +\infty} p(x) = +\infty$, any solution $y(x)$ of (L) is oscillatory, in other words vanishes for arbitrarily large values of x. To show that $y(x)$ is bounded, it is therefore sufficient to prove that the absolute values of $y(x)$ at its relative maximum and minimum points in $[b, \infty)$ are all less than some fixed bound. Suppose then that the values are unbounded. Then there exists a sequence $\{c_n\}$ such that,

(5) $$c_n > b \quad \lim_{n \to \infty} c_n = +\infty,$$

(6) $$y'(c_n) = 0,$$

(7) $$|y(c_n)| = \max \{|y(x)| \mid x \in [b, c_n]\},$$

(8) $$\lim_{n \to \infty} |y(c_n)| = +\infty.$$

Now (1), (3), (5), and (6) imply the inequality

$$2[y(c_n)]^2 < \left[\sqrt{p(x)} + \left(\frac{1}{\sqrt{p(x)}}\right)''\right]_{x=c_n} [y(c_n)]^2$$
$$= w(a) + \int_a^{c_n} \left(\frac{1}{\sqrt{p(x)}}\right)''' \frac{[y(x)]^2}{2}\, dx.$$

Hence,

$$2[y(c_n)]^2 \leq \left| w(a) + \int_a^b \left(\frac{1}{\sqrt{p(x)}}\right)''' \frac{[y(x)]^2}{2}\, dx \right| + \int_b^{c_n} \left|\left(\frac{1}{\sqrt{p(x)}}\right)'''\right| \frac{[y(x)]^2}{2}\, dx.$$

Since (7) implies that $|y(c_n)| \geq |y(x)|$ for $x \in [b, c_n]$, we obtain the further inequality

$$2[y(c_n)]^2 \leq \left| w(a) + \frac{1}{2} \int_a^b \left(\frac{1}{\sqrt{p(x)}}\right)''' [y(x)]^2\, dx \right| + \frac{[y(c_n)]^2}{2} \int_b^{c_n} \left|\left(\frac{1}{\sqrt{p(x)}}\right)'''\right| dx;$$

and because the second integral in the right-hand member is less than 1, by (4), it follows that

$$\frac{3}{2}[y(c_n)]^2 < [y(c_n)]^2\left[2 - \frac{1}{2}\int_b^{c_n}\left|\left(\frac{1}{\sqrt{p(x)}}\right)'''\right| dx\right]$$

$$\leq \left| w(a) + \frac{1}{2}\int_a^b \left(\frac{1}{\sqrt{p(x)}}\right)''' [y(x)]^2\, dx\right|.$$

The last member is independent of n, and therefore the sequence $\{y(c_n)\}$ is bounded, contrary to (8). This contradiction shows that $\{y(c_n)\}$ is indeed bounded.

To complete the proof that $\lim_{x\to+\infty} y(x) = 0$, let M be a constant such that $|y(x)| < M$ for all $x \in [a, \infty)$.

From (1) and the assumption $\int_a^\infty \left|\left(\frac{1}{\sqrt{p(x)}}\right)'''\right| dx < +\infty$, it follows that

$$w(x) = \frac{[y'(x)]^2}{\sqrt{p(x)}} - \left(\frac{1}{\sqrt{p(x)}}\right)' y(x)\,y'(x) + \left[\sqrt{p(x)} + \frac{1}{2}\left(\frac{1}{\sqrt{p(x)}}\right)''\right][y(x)]^2$$

$$(9) \qquad = w(a) + \int_a^x \left(\frac{1}{\sqrt{p(t)}}\right)''' \frac{[y(t)]^2}{2}\, dt$$

$$< |w(a)| + \frac{M}{2}\int_a^\infty \left|\left(\frac{1}{\sqrt{p(t)}}\right)'''\right| dt \equiv c^2 < +\infty.$$

Let $x_1 < x_2 < x_3 < \cdots$ be the successive relative maximum and minimum points of the solution y(x), which we may assume to be nontrivial. Then

$$(10) \qquad y'(x_n) = 0, \qquad \lim_{n\to\infty} x_n = +\infty,$$

and by (9),

$$w(x_n) = \left[\sqrt{p(x)} + \frac{1}{2}\left(\frac{1}{\sqrt{p(x)}}\right)''\right]_{x=x_n} [y(x_n)]^2 < c^2.$$

From (2) and (10) we deduce that

$$\lim_{n\to\infty}\left[\sqrt{p(x)} + \frac{1}{2}\left(\frac{1}{\sqrt{p(x)}}\right)''\right]_{x=x_n} = +\infty$$

and hence $\lim_{n\to\infty} |y(x_n)| = 0$. From this we see at once that $\lim_{x\to+\infty} |y(x)| = 0$.

REFERENCE

1. L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, Second Edition, Springer Verlag, Berlin, 1963.

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