AN EXTENSION OF LYAPUNOV'S DIRECT METHOD

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1. INTRODUCTION

Corduneanu [3] and Antosiewicz [1] observed that the direct method of Lyapunov depends basically on the fact that a function $m(t)$ satisfying the inequality $\dot{m}(t) \leq w(t, m(t))$ $(m(t_0) \leq r_0)$ is majorized by the maximal solution of the scalar differential equation $\dot{r} = w(t, r)$, $r(t_0) = r_0$. Lakshmikantham [4, 5] and others have made consistent use of this remark to extend the direct method of Lyapunov to various stability and boundedness problems.

Stability and boundedness are almost exclusively defined in terms of a distance from a given point. However, Ling [6] defined stability with respect to a manifold. In this paper we define stability and boundedness with respect to a manifold in a way more general than that of Ling. Then, by comparison with a scalar differential equation (as in previous papers of Lakshmikantham), we obtain theorems of boundedness and stability with respect to a manifold.

2. NOTATION AND DEFINITIONS

Let I denote the half-line $0 \leq t < +\infty$, and let R^n denote n-dimensional real euclidean space. We consider the system

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0 \qquad (t_0 \geq 0), \tag{2.1}$$

where x and f are n-vectors, where the function $f(t, x)$ is defined and continuous on the product space $I \times R^n$, and where $(\dot{\ }) = d/dt$.

Let g be a k-dimensional vector $(k \leq n)$, and suppose that the function $g(t, x)$ is defined and continuous on the product space $I \times R^n$. For each $t \in I$ let the set of points x satisfying the relation

$$g(t, x) = 0 \tag{2.2}$$

define an $(n - k)$-manifold $M_t(n - k)$.

Define $\|g(t, x)\|^2 = \sum_{i=1}^{k} g_i^2(t, x)$. For each $t \in I$ denote the sets

$$\{x: \|g(t, x)\| < \eta\} \qquad \text{and} \qquad \{x: \|g(t, x)\| \leq \eta\}$$

by $M_t(n - k)(\eta)$ and $M_t(n - k)(\bar{\eta})$, respectively. Suppose that $x(t)$ is any solution of the system (2.1).

In order to unify our results on stability and boundedness of the system (2.1) with respect to functions g satisfying (2.2), we list the following conditions:

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(2.3) There exists a positive function $\delta = \delta(t_0, \eta)$ $(t_0 \geq 0,\ \eta > 0)$, continuous in t_0 for each η, such that

$$x_0 \in M_t(n-k)(\bar{\delta}) \quad \text{implies} \quad x(t) \in M_t(n-k)(\eta) \text{ for } t \geq t_0,$$

(2.4) the function δ is (2.3) is independent of t_0,

(2.5) for each $\varepsilon > 0$, $\alpha > 0$, $t_0 \geq 0$, there exists a positive number $T = T(t_0, \varepsilon, \alpha)$ such that

$$x_0 \in M_{t_0}(n-k)(\bar{\alpha}) \quad \text{implies} \quad x(t) \in M_t(n-k)(\varepsilon) \text{ for } t \geq t_0 + T,$$

(2.6) the number T in (2.5) is independent of t_0,

(2.7) for each $\alpha > 0$ and $t_0 \geq 0$, there exists a positive function $\beta = \beta(t_0, \alpha)$, continuous in t_0 for each α, such that

$$x_0 \in M_{t_0}(n-k)(\bar{\alpha}) \quad \text{implies} \quad x(t) \in M_t(n-k)(\beta) \text{ for } t \geq t_0,$$

(2.8) the function β in (2.7) is independent of t_0,

(2.9) for each $\alpha > 0$ and $t_0 \geq 0$ there exist positive numbers $T = T(t_0, \alpha)$ and $N = N(t_0)$ such that

$$x_0 \in M_{t_0}(n-k)(\bar{\alpha}) \quad \text{implies} \quad x(t) \in M_t(n-k)(N) \text{ for } t \geq t_0 + T,$$

(2.10) the numbers N and T in (2.9) are independent of t_0.

Remark 2.1. Clearly if condition (2.3) is satisfied, then $x_0 \in M_{t_0}(n-k)$ implies that $x(t) \in M_t(n-k)$ for $t \geq t_0$. Thus the subset $\{(t, x): t \geq 0,\ x \in M_t(n-k)\}$ of $I \times R^n$ is a positively invariant set of the system (2.1).

Remark 2.2. If $n = k$ and $g(t, x) = x$, our conditions are reduced to ordinary stability of an equilibrium point (the origin).

Remark 2.3. The conditions introduced by Ling [6] correspond to our conditions (2.4) and (2.6), when $g(t, x)$ is independent of t.

Remark 2.4. Condition (2.3) can also be formulated as follows:

(2.3)' for each $\eta > 0$ and $t_0 \geq 0$, there exists a positive function $\delta(t_0, \eta)$, continuous in t_0 for each η, such that

$$\|g(t_0, x_0)\| \leq \delta \quad \text{implies} \quad \|g(t, x(t))\| < \eta \text{ for } t \geq t_0.$$

One may similarly reformulate the remaining conditions (2.4) to (2.10).

3. THE BASIC LEMMAS

Our results on stability and boundedness depend on two lemmas proved elsewhere [4], [5]. We shall state them here in a suitable form.

Let the function $V(t, x) \geq 0$ be defined and continuous on the product space $I \times R^n$. Suppose that for each $t \in I$, $V(t, x) = 0$ if $x \in M_t(n - k)$. Let $V(t, x)$ satisfy a local Lipschitz condition in x. Define the function

(3.1) $$V^* = V^*(t, x) = \limsup_{h \to 0+} \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)].$$

Let R^+ denote the interval $[0, +\infty)$.

LEMMA 3.1. *Let the function* $w(t, r)$ *be defined and continuous on* $I \times R^+$. *Suppose further that the function* $V^*(t, x)$ *of* (3.1) *satisfies the condition*

(3.2) $$V^*(t, x) \leq w(t, V(t, x)).$$

Let $r(t)$ *be the maximal solution of the scalar differential equation*

(3.3) $$\dot{r} = w(t, r), \qquad r(t_0) = r_0,$$

existing to the right of t_0. *If* $x(t)$ *is any solution of* (2.1) *such that*

(3.4) $$V(t_0, x_0) \leq r_0,$$

then

(3.5) $$V(t, x(t)) \leq r(t) \qquad \text{for } t \geq t_0 .$$

LEMMA 3.2. *Suppose that the assumptions of Lemma* 3.1 *hold, except that condition* (3.2) *is replaced by*

(3.6) $$A(t)\, V^*(t, x) + A^*(t)\, V(t, x) \leq w(t, A(t)\, V(t, x)),$$

where the function $A(t)$ *is defined, continuous, and positive on* I, *and where*

$$A^*(t) = \limsup_{h \to 0+} \frac{1}{h} [A(t + h) - A(t)].$$

Then the condition

(3.7) $$A(t_0)\, V(t_0, x_0) \leq r_0$$

implies that

(3.8) $$A(t)\, V(t, x(t)) \leq r(t) \qquad \textit{for } t \geq t_0 .$$

Henceforth we assume that the solutions $r(t)$ of (3.3) are nonnegative for $t \geq t_0$, so as to ensure that $w(t, r(t))$ is defined. Such a requirement is clearly satisfied if we assume that $w(t, 0) = 0$ for $t \geq 0$.

Remark 3.1. Corresponding to condition (2.3), we say that the differential equation (3.3) has the property (2.3s) provided the following condition is satisfied:

(2.3s) There exists a positive function $\delta = \delta(t_0, \eta)$ ($t_0 \geq 0$, $\eta > 0$), continuous in t_0 for each η, such that $r(t) < \eta$ for $t \geq t_0$ whenever $r_0 \leq \delta$.

Conditions (2.4) to (2.10) may similarly be reformulated into conditions (2.4s) to (2.10s) for the differential equation (3.3).

4. THE V-FUNCTION

In order to formulate theorems on stability and boundedness defined by conditions (2.3) to (2.10) for the system (2.1), we shall require the function $V(t, x)$ of Section 3 to satisfy one or more of the following conditions:

(4.1) There exist two continuous strictly increasing functions $a(r)$ and $b(r)$ defined for $r \geq 0$, and a positive continuous function $\gamma(t) \geq 1$, defined for $t \geq 0$, such that

$$a(\|g(t, x)\|) \leq V(t, x) \leq \gamma(t)\, b(\|g(t, x)\|), \qquad a(0) = b(0) = 0,$$

(4.2) $$\gamma(t) \equiv 1 \quad \text{in (4.1)},$$

(4.3) $$a(r) \to +\infty \quad \text{as } r \to \infty.$$

Remark 4.1. We recall that a continuous function $V(t, x)$ with $V(t, 0) = 0$ for $t \geq 0$ is positive definite if and only if $a(\|x\|) \leq V(t, x)$, where $a(r)$ is a strictly increasing continuous function and $a(0) = 0$. We see that in this case $V(t, x)$ also satisfies an inequality $V(t, x) \leq \gamma(t) b(\|x\|)$, where $\gamma(t)$ and $b(r)$ are as defined in (4.1). This becomes clear if we set

$$\gamma(t) = \sup_{\|x\| \leq t} V(t, x) + 1 \qquad \text{and} \qquad b(r) = \sup_{\substack{t \leq 0 \\ \|x\| \leq r}} \frac{V(t, x)}{\gamma(t)}.$$

In the simple case $g(t, x) = x$, (4.1) is thus equivalent to the statement that $V(t, x)$ is positive definite. We remark further that in our general case the condition $a(\|g(t, x)\|) \leq V(t, x)$ does not necessarily imply (4.1).

5. STABILITY AND BOUNDEDNESS THEOREMS

We now state our main results.

THEOREM 5.1. *Let the assumptions of Lemma* 3.1 *hold, together with* (4.1), *and let the differential equation* (3.3) *satisfy* (2.3s) *or* (2.5s); *then the system* (2.1) *satisifes the corresponding condition* (2.3) *or* (2.5).

THEOREM 5.2. *Let the assumptions of Lemma* 3.1 *hold, together with* (4.2), *and let the differential equation* (3.3) *satisfy one of the conditions* (2.3s), (2.4s), (2.5s), *and* (2.6s); *then the system* (2.1) *satisfies the corresponding one of the conditions* (2.3), (2.4), (2.5), *and* (2.6).

THEOREM 5.3. *Let the assumptions of Lemma* 3.1 *hold, together with* (4.1) *and* (4.3), *and let the differential equation* (3.3) *satisfy condition* (2.7s) *or* (2.9s); *then the system* (2.1) *satisfies the corresponding condition* (2.7) *or* (2.9).

THEOREM 5.4. *Let the assumptions of Lemma* 3.1 *hold, together with* (4.2) *and* (4.3), *and let the differential equation* (3.3) *satisfy one of the conditions* (2.7s), (2.8s), (2.9s), *and* (2.10s); *then the system* (2.1) *satisfies the corresponding one of the conditions* (2.7), (2.8), (2.9), *and* (2.10).

THEOREM 5.5. *Let the assumptions of Lemma* 3.2 *hold, together with* (4.1); *further let* $A(t) \to +\infty$ *as* $t \to +\infty$, *and let the differential equation* (3.3) *satisfy condition* (2.3s); *then the system* (2.1) *satisfies conditions* (2.3) *and* (2.5).

THEOREM 5.6. *Let the assumptions of Lemma* 3.2 *hold, together with* (4.1) *and* (4.3), *let* $A(t) \to \infty$ *as* $t \to \infty$, *and let the differential equation* (3.3) *satisfy* (2.7s); *then the system* (2.1) *satisfies conditions* (2.7) *and* (2.9).

If $a(r)$ and $b(r)$ are defined as in (4.1), we shall denote their inverse functions by $a^{-1}(r)$ and $b^{-1}(r)$. Note that they are continuous and strictly increasing, and that $a^{-1}(0) = b^{-1}(0) = 0$.

Proof of Theorem 5.1. Suppose first that the scalar differential equation (3.3) satisfies (2.3s). For any solution $x(t)$, $x(t_0) = x_0$, of the system (2.1), (3.4) implies (3.5) by Lemma 3.1. If V satisfies (4.1), then (3.4) holds if

$$\gamma(t_0)\, b(\|g(t_0, x_0)\|) \leq r_0 .$$

Setting now $r_0 = \delta(t_0, \eta)$, we see that

$$\gamma(t_0)\, b(\|g(t_0, x_0)\|) \leq \delta(t_0, \eta)$$

implies

$$a(\|g(t, x(t))\|) \leq V(t, x(t)) \leq r(t) < \eta \qquad \text{for } t \geq t_0 .$$

In other words

$$\|g(t_0, x_0)\| \leq b^{-1}\left(\frac{\delta(t_0, \eta)}{\gamma(t_0)}\right) \tag{5.1}$$

implies

$$\|g(t, x(t))\| \leq a^{-1}(\eta). \tag{5.2}$$

If for any $\eta^* > 0$ and $t_0 \geq 0$ we set

$$\eta = a(\eta^*) \qquad \text{and} \qquad \delta^*(t_0, \eta^*) = b^{-1}[\delta(t_0, \eta)/\gamma(t_0)],$$

we can conclude that the system (2.1) satisfies (2.3).

Now suppose that the scalar differential equation (3.3) satisfies (2.5s). For any solution $x(t)$, $x(t_0) = x_0$, of the system (2.1) we notice that (3.4) implies (3.5). If V satisfies (4.1), then (3.4) is indeed satisfied if

$$\gamma(t_0)\, b(\|g(t_0, x_0)\|) \leq r_0 .$$

Setting $\alpha = r_0$, we see that

$$\gamma(t_0)\, b(\|g(t_0, x_0)\|) \leq \alpha \qquad \text{implies} \qquad a(\|g(t, x(t))\|) \leq V(t, x(t)) \leq r(t) < \varepsilon$$

for $t \geq t_0 + T(t_0, \varepsilon, \alpha)$. In other words,

$$\|g(t_0, x_0)\| \leq b^{-1}[\alpha/\gamma(t_0)] \tag{5.3}$$

implies

(5.4) $$\|g(t, x(t))\| < a^{-1}(\varepsilon) \quad \text{for } t \geq t_0 + T(t_0, \varepsilon, \alpha).$$

We see now that for any $\varepsilon^* > 0$, $\alpha^* > 0$, and $t_0 \geq 0$, we can set

$$\varepsilon = a(\varepsilon^*), \qquad \alpha = \gamma(t_0) b(\alpha^*), \qquad T^*(t_0, \varepsilon^*, \alpha^*) = T(t_0, \varepsilon, \alpha)$$

to conclude that the system (2.1) satisfies (2.5). This completes the proof of the theorem.

Proof of Theorem 5.2. The proof of this theorem is entirely analogous to that of Theorem 5.1. One needs to set $\gamma(t_0) \equiv 1$ and to observe that if the differential equation (3.3) satisfies (2.4s) or (2.6s), the functions δ^* and T^*, are, respectively, independent of t_0.

Proof of Theorem 5.3. Suppose first that the scalar differential equation (3.3) satisfies (2.7s). Condition (3.4) implies (3.5) by Lemma 3.1, and if V satisfies (4.1), then (3.4) holds whenever

$$\gamma(t_0) b(\|g(t_0, x_0)\|) \leq r_0.$$

Setting $r_0 = \alpha$, we see that

$$\gamma(t_0) b(\|g(t_0, x_0)\|) \leq \alpha \quad \text{implies} \quad a(\|g(t, x(t))\|) \leq V(t, x(t)) \leq r(t) < \beta(t_0, \alpha)$$

for $t \geq t_0$. In other words,

(5.5) $$\|g(t_0, x_0)\| \leq b^{-1}[\alpha/\gamma(t_0)]$$

implies

(5.6) $$\|g(t, x(t))\| < a^{-1}(\beta(t_0, \alpha)) \quad \text{for } t \geq t_0.$$

If now $\alpha^* > 0$ and $t_0 \geq 0$, we need only set

$$\alpha = \gamma(t_0) b(\alpha^*) \quad \text{and} \quad \beta^*(t_0, \alpha^*) = a^{-1}(\beta(t_0, \alpha))$$

to see that the system (2.1) satisfies (2.7) if (4.3) holds. Note that we require the condition (4.3) to ensure that $\alpha^* \to \infty$ as $\alpha \to \infty$.

Suppose now that the differential equation (3.3) satisfies (2.9s). Again (3.4) implies (3.5) by Lemma 3.1, and if V satisfies (4.1), then (3.4) holds if

$$\gamma(t_0) b(\|g(t_0, x_0)\|) \leq r_0.$$

Setting $r_0 = \alpha$, we then conclude that

$$\gamma(t_0) b(\|g(t_0, x_0)\|) \leq \alpha \quad \text{implies} \quad a(\|g(t, x(t))\|) \leq V(t, x(t)) \leq r(t) < N(t_0)$$

for $t \geq t_0 + T(t_0, \alpha)$. In other words,

(5.7) $$\|g(t_0, x_0)\| \leq b^{-1}[\alpha/\gamma(t_0)]$$

implies

(5.8) $$\|g(t, x(t))\| < a^{-1}(N(t_0)) \quad \text{for } t \geq t_0 + T(t_0, \alpha).$$

Given any $\alpha^* > 0$ and $t_0 \geq 0$, we need only set

$$\alpha = \gamma(t_0)\, b(\alpha^*), \qquad N^*(t_0) = a^{-1}(N(t_0)), \qquad T^*(t_0, \alpha^*) = T(t_0, \alpha)$$

to note that the system (2.1) satisfies (2.9) provided (4.3) holds. The last requirement is needed to ensure that $\alpha \to \infty$ as $\alpha^* \to \infty$. This completes the proof of Theorem 5.3.

Proof of Theorem 5.4. The proof of this theorem is analogous to that of Theorem 5.3. One need only set $\gamma(t_0) \equiv 1$ and observe that β^*, T^*, and N^* are independent of t_0 if (3.3) satisfies one of the conditions (2.8) and (2.10).

Proof of Theorem 5.5. Let $m = \inf_{t \geq 0} A(t)$. Then $m > 0$, by our assumptions on $A(t)$. If V satisfies (4.1), then

$$\text{(5.9)} \qquad m a(\| g(t, x) \|) \leq A(t) a(\| g(t, x) \|) \leq A(t) V(t, x) \leq A(t) \gamma(t) b(\| g(t, x) \|).$$

Let now (3.3) satisfy (2.3s), and let Lemma 3.2 hold, so that (3.7) implies (3.8). Condition (3.7) will be satisfied if we require that

$$A(t_0)\gamma(t_0) b(\| g(t_0, x_0) \|) \leq r_0.$$

Setting now $r_0 = \delta(t_0, \eta)$, we see that because of (5.9) the condition

$$A(t_0)\gamma(t_0) b(\| g(t_0, x_0) \|) \leq \delta(t_0, \eta)$$

implies

$$m a(\| g(t, x(t)) \|) \leq A(t) a(\| g(t, x(t)) \|) \leq A(t) V(t, x(t)) \leq r(t) < \eta$$

for $t \geq t_0$. In other words

$$\text{(5.10)} \qquad \| g(t_0, x_0) \| \leq b^{-1}[\delta(t_0, \eta)/A(t_0)\gamma(t_0)]$$

implies

$$\text{(5.11)} \qquad \| g(t, x(t)) \| < a^{-1}(\eta/m) \quad \text{for } t \geq t_0$$

and

$$\text{(5.12)} \qquad \| g(t, x(t)) \| < a^{-1}(\eta/A(t)) \quad \text{for } t \geq t_0.$$

Given now any $\eta^* > 0$ and $t_0 \geq 0$, we set

$$\eta = m a(\eta^*) \quad \text{and} \quad \delta^*(t_0, \eta^*) = b^{-1}[\delta(t_0, \eta)/A(t_0)\gamma(t_0)]$$

in (5.10) and (5.11), to see that the system (2.1) satisfies (2.3). Again, given $\alpha > 0$, $\varepsilon > 0$, and $t_0 \geq 0$, we see that $\eta > 0$ can be fixed so that

$$\alpha = b^{-1}[\delta(t_0, \eta)/A(t_0)\gamma(t_0)].$$

Since $\eta/A(t) \to 0$ as $t \to \infty$, one can choose $T = T(\eta, \varepsilon)$ so that

$$\| g(t, x(t)) \| < \varepsilon \quad \text{for } t \geq T(\eta, \varepsilon).$$

Setting $T^*(t_0, \varepsilon, \alpha) = T(\eta, \varepsilon) - t_0$ and noting that η depends on α and t_0, we conclude that the system (2.1) satisfies (2.5). The proof of Theorem 5.5 is now complete.

The proof of Theorem 5.6 is similar to the proof of Theorem 5.5, except that one needs condition (4.3). We leave it to the reader.

Remark 5.1. The functions a(r) and b(r) in conditions (4.1) and (4.2) need only be assumed nondecreasing and positive for $r > 0$. Since, however, this implies the existence of strictly increasing functions satisfying (4.1) and (4.2), the conditions are not effectively weakened.

Remark 5.2. Theorem 4 in [4] is erroneous. Corrections can be made in the light of our Theorems 5.1 and 5.2 above. We give a simple example to illustrate our point:

Example 5.1. Let $g(t, x) = x$, and consider the scalar differential equation

$$\dot{x} = [\sin \log t + \cos \log t - a]x, \tag{5.13}$$

whose general solution is

$$x = x_0 \exp[t(\sin \log t - a) - t_0(\sin \log t_0 - a)]. \tag{5.14}$$

One can easily verify that if $1 \leq a < \sqrt{2}$, then (5.13) satisfies (2.3) but not (2.4). Choose

$$V = x^2 \exp[2(a - \sin \log t)t].$$

If $1 \leq a$, then V satisfies (4.1) but not (4.2), and $V^* = 0$. The scalar equation $\dot{r} = 0$ satisfies (2.4s) and thus also (2.3s). We can conclude from Theorem 5.1 that (5.13) satisfies (2.3). In fact, Theorem 5.5 is applicable and shows that (5.13) satisfies (2.3) and (2.5). However, if Lakshmikantham's Theorem 4 in [4] were applied to this situation, one would conclude that (5.13) satisfies (2.4), which is not true if $1 \leq a < \sqrt{2}$. Further, the conditions of Theorems 5 and 6 in [4] are also fulfilled by our example. If all were well, this would imply that (5.13) satisfies (2.4) and (2.6), which is not true when $1 < a < \sqrt{2}$, since in this case (5.13) satisfies only (2.3) and (2.5). Similar remarks apply to Theorem 3 in [4] and Theorems 4, 5, 6, and 7 in [5]. However, in the light of our present work the formulations and proofs of these theorems can be corrected.

Remark 5.3. We notice also that the condition $A(t) \geq 1$ imposed by Lakshmikantham in his Theorems 5 and 6 in [4] and Theorems 6 and 7 in [5] is not required, as our theorems 5.5 and 5.6 show.

We now give two applications of our theorems.

Example 5.2. Consider the system

$$\dot{x} = x(2 + y)\sin t, \qquad \dot{y} = (2ax + y)\sin t.$$

Let $g = y^2 - 4ax$ and $V = g^2$. Then (4.2) and (4.3) are clearly satisfied, and

$$V^* = 4(y^2 - 4ax)^2 \sin t = 4V \sin t.$$

Taking $w(t, r) = 4r \sin t$, we see that (3.3) satisfies (2.3s) and (2.7s). From Theorems 5.2 and 5.4 we conclude therefore that the system satisfies (2.3) and (2.7).

Example 5.3. Consider the system

$$\dot{x} = -2x[y^2 - (2+\sin t)x]^2 - x\cos t/(2+\sin t),$$

$$\dot{y} = -y[y^2 - (2+\sin t)x]^2.$$

Let $g = y^2 - (2+\sin t)x$ and $V = g^2$, so that (4.2) and (4.3) are satisfied.

A simple calculation shows that $V^* = -4g^4 = -4V^2$. If we set $w(t, r) = -4r^2$, we see that (3.3) satisfies (2.4s), (2.6s), (2.8s), and (2.10s). By Theorems 5.2 and 5.4, the system satisfies (2.4), (2.6), (2.8), and (2.10).

REFERENCES

1. H. A. Antosiewicz, *An inequality for approximate solutions of ordinary differential equations*, Math. Z. 8 (1962), 44-52.
2. N. P. Bhatia, *Stability and Lyapunov functions in dynamical systems*, Contributions to Differential Equations 3 (1964), 175-188.
3. C. Corduneanu, *The application of differential inequalities to the theory of stability*, (Russian), An. Şti. Univ. "Al. I. Cuza" Iaşi, Secţ. I N. S. 6 (1960), 47-58.
4. V. Lakshmikantham, *Differential systems and extension of Lyapunov's method*, Michigan Math. J. 9 (1962), 311-320.
5. ———, *Differential equations in Banach spaces and the extension of Lyapunov's method*, Proc. Cambridge Philos. Soc. 59 (1963), 373-381.
6. H. Ling, *On the estimation of the decaying time*, Proceedings of the International Federation of Automatic Control Congress, 1963, Basle, Switzerland; Butterworth, London (to appear).

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