COMPLETE SYSTEMS IN L_2 AND A THEOREM OF RÉNYI

Richard F. Gundy

In [4] Rényi refers to a question and a conjecture of H. Steinhaus about families $S = \{f_n(x)\}_{n=1}^{\infty}$ of stochastically independent, bounded Borel functions defined on the unit interval. From such a family S, we may form the family S^* of all finite products of powers of elements of S:

$$S^* = \left\{ \prod_{k=1}^{N} f_{n_k}^{m_k}; \; n_k, \, m_k, \, N \text{ arbitrary integers} \right\}.$$

What are necessary and sufficient conditions on the family S *in order that the family* S^* *be complete in* $L^2(0, 1)$, *in the sense that the only functions orthogonal to every member of* S^* *vanish almost everywhere?* Steinhaus conjectured that if no nonconstant function, stochastically independent of every member of S, can be defined on the unit interval I, then S^* is complete. Systems for which no nonconstant independent functions exist are said to be *saturated with respect to independence.* This property is easily seen to be necessary for completeness of S^*, and the conjectured sufficiency is supported by familiar examples. One of these is the system of Rademacher functions; here S^* is the system of Walsh functions, known to be complete. Another example is the system

$$S = \{f_1(x) = 1, \; f_2(x) = x\}, \qquad S^* = \{x^n; \, n = 0, 1, \cdots\},$$

which is again known to be complete.

Rényi [4] considers a more general problem, in which the members of S are not necessarily stochastically independent. In this setting, he shows that saturation with respect to independence is not sufficient for completeness of S^*. In fact, if S consists of the single function

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1/2, \\ 1 & \text{if } 1/2 < x \le 1, \end{cases}$$

then S^* is saturated with respect to independence, yet any L^2-function vanishing on $[0, 1/2)$ and odd about $3/4$ on $[1/2, 1]$ is orthogonal to S^*. However, Rényi gives the following sufficient condition for completeness of S^* [4].

THEOREM (Rényi). *If* $S = \{f_n\}_{n=1}^{\infty}$ *is a family of bounded Borel functions on the unit interval* I, *and if* I *contains a set* M *of Lebesgue measure* 1 *such that for each pair* x, y *with* $x \in M$, $y \in M$, *and* $x \ne y$ *the inequality* $f_n(x) \ne f_n(y)$ *holds for some* n, *then the corresponding family* S^* *is complete.*

This theorem may be applied to prove completeness for many classical systems of orthogonal functions, including the Walsh and trigonometrical systems.

Received July 29, 1964.
This research was supported by NSF Contract GP - 281.

In the first part of this paper we consider the general problem proposed by Rényi, and we give a necessary and sufficient condition for completeness of S^*. From this, it follows that Rényi's condition is necessary as well as sufficient for completeness.

At the conclusion of his paper, Rényi remarks that although the original Steinhaus question has a negative answer, it would be of interest to find necessary and sufficient conditions for completeness of S^* when S is an infinite family of independent functions. This problem is considered in the second section of this paper, under the additional hypothesis that S is a family of independent binomial functions, a specialization suggested by the Rademacher system. We show that a system S of independent binomial variables whose distributions are given by a sequence

$$\{p_n\}_{n=1}^{\infty} \quad \text{where } 0 < p_n \leq 1/2$$

may be constructed so that S^* is complete if and only if $\Sigma p_n = \infty$.

It is still conceivable that saturation with respect to independence is sufficient for completeness of S^*, if S is suitably restricted. An appropriate restriction seems difficult to formulate, even when S consists of independent binomial functions. In fact, we give an example of a family S with the following properties: a) S is a sequence of independent symmetric binomial functions; b) S is saturated with respect to independence; c) S^* is incomplete.

In the third section, families of functions that are saturated with respect to independence are characterized in terms of the associated family of conditional probability distributions.

1. A NECESSARY AND SUFFICIENT CONDITION FOR COMPLETENESS

If $S = \{f_n\}_1^{\infty}$ is a sequence of functions, we denote by $\sigma(S)$ the smallest σ-field with respect to which all members of S are measurable, and we say that $\sigma(S)$ is *generated* by S. Two σ-fields will be called *equivalent* if they are identical up to sets of measure zero. For any $f \in L^2(0, 1)$ and any σ-field $\sigma(S)$, we denote by $E(f \parallel \sigma(S))$ the conditional expectation of f relative to $\sigma(S)$.

In the following, the underlying space will be the unit interval I with Lebesgue measure on the σ-field β of Borel sets. (This restriction is a matter of convenience, not entirely necessary for the validity of the results.) If A is a Borel set, then $|A|$ will denote the Lebesgue measure A.

We say that a set A is an *atom* in a σ-field $\sigma(S)$ if $A \in \sigma(S)$, $|A| > 0$, and $|E| = 0$ or $|E| = |A|$ whenever $E \in \sigma(S)$ and $E \subset A$.

THEOREM 1. *If* S *is a family of bounded Borel measurable functions, then a necessary and sufficient condition that* S^* *be complete in* $L^2(0, 1)$ *is that* $\sigma(S)$ *be equivalent to the Borel sets of* $(0, 1)$.

Necessity. If $\sigma(S)$ is not equivalent to the Borel sets, then there exists a Borel function $f(x) \in L^2(0, 1)$ such that

$$f(x) - E(f \parallel \sigma(S))(x) \neq 0$$

on a set of positive measure. Since $\sigma(S) = \sigma(S^*)$

$$E(f \parallel \sigma(S)) = E(f \parallel \sigma(S^*))$$

almost everywhere, so that

$$h(x) = f(x) - E(f \parallel \sigma(S^*))(x) \neq 0$$

on a set of positive measure. The function $h(x)$ is orthogonal to every member of S^*.

Sufficiency. It will be shown that there exists an L^2-norm approximation to an arbitrary function $g(x) \in L^2(0, 1)$ by a finite linear combination of elements from S^*. Let $\sigma(S_N)$ be the σ-field generated by $S_N = \{f_n\}_1^N$. The hypothesis of the theorem guarantees that $E(g \parallel \sigma(S)) = g$ almost everywhere. The martingale convergence theorem then implies that

$$\int |g - E(g \parallel \sigma(S_N))|^2 \, dx \leq \varepsilon \qquad \text{for } N \geq N_\varepsilon .$$

We shall show that the function $E(g \parallel \sigma(S_N))$ can be approximated arbitrarily closely in the L^2-norm by a finite linear combination of elements from S^*. If we combine this fact with the above inequality, the proof is complete. The approximation problem for a general L^2-function of the form $E(g \parallel \sigma(S_N))$ may be reduced by a familiar argument to finding approximations for characteristic functions of sets of the form $\bigcap_{k=1}^M \{f_{n_k} > a_k\}$. Since $|f_n| \leq C(N)$ almost everywhere for $n = 1, 2, \cdots, N$, it is possible to find an L^2-norm approximation for each function $|f_n(x)|$ by a polynomial in $f_n(x)$. It follows from this that the following functions have polynomial approximations in the L^2-norm.

(a) $f_n^+ = (|f_n| + f_n)/2$,

(b) $\min(f_n, 1) = (f_n^+ + 1 - |f_n^+ - 1|)/2$,

(c) the characteristic function $\chi(f_n > 0)(x)$ of the set

$$\{x: f_n(x) > 0\} = \lim_{k \to \infty} \min(k \cdot \min(f_n^+, 1), 1).$$

Since the above statements hold as well for $f_n(x) - a$, it follows that the characteristic function of $\{f_n > a\}$ has a polynomial approximation. If we choose the degree of approximation appropriately, we can multiply the polynomials for $\{f_{n_k} > a_{n_k}\}$, forming a linear combination of elements of S^*, to give an approximation for the characteristic function of the set $\bigcap_{k=1}^M \{f_{n_k} > a_k\}$. As pointed out above, a familar argument leads from this to an approximation of simple functions, and ultimately to an approximation of the general L^2-function. This completes the proof of Theorem 1.

Rényi's sufficient condition for completeness of S^*, quoted above, is also necessary; the proof of this follows from Theorem 1.

COROLLARY. *Rényi's condition is necessary and sufficient for completeness of* S^*.

Suppose S^* is complete. Then it follows from Theorem 1 that $\sigma(S)$ is equivalent to β. If $\sigma(S_N)$ is the σ-field generated by $\{f_n(x)\}_1^N$ and e is the identity $e(x) = x$, then

$$\lim_{N\to\infty} E(e \,\|\, \sigma(S_N)) = \lim_{N\to\infty} h_N(f_1, \cdots, f_N) = e$$

on a set M of measure 1, where each h_N is an appropriately chosen Borel function on E^N. If x, y $\in$ M and x $\neq$ y, it follows that

$$h_N(f_1(x), \cdots, f_N(x)) \neq h_N(f_1(y), \cdots, f_N(y))$$

for some N. This implies $f_n(x) \neq f_n(y)$ for some n $(1 \leq n \leq N)$, so that Rényi's condition is in fact necessary as well as sufficient for completeness of S^*.

2. INDEPENDENT BINOMIAL FUNCTIONS

Returning to the original problem of Steinhaus, suppose that $S = \{f_n\}_1^\infty$ is a system of nonconstant independent functions. Necessary and sufficient conditions for completeness of S^* that utilize the independence hypothesis in an essential way seem difficult to obtain. The following considerations may point out the nature of the difficulty. Suppose S is a family of independent binomial functions defined on the unit interval; that is, let each f_n take on only two values, say ± 1, with probabilities p_n and q_n $(p_n \leq q_n,\ p_n + q_n = 1)$, so that the class of all such systems contains the system of Rademacher functions as a member.

Denote by $\{S, p_n\}_1^\infty$ the class of all binomial systems having the associated sequence $\{p_n\}$ of probabilities. For example, the Rademacher system belongs to the class $\{S, p_n \equiv 1/2\}$.

THEOREM 2. *Each class* $\{S, p_n\}$ *of independent binomial systems contains a member* S *such that* S^* *is complete if and only if* $\Sigma\, p_n = \infty$.

Proof. We shall show that there exists a member $S \in \{S, p_n\}$ with the property that $\sigma(S)$ is equivalent to the Borel sets if and only if $\Sigma\, p_n = \infty$. The proof will then be completed by an appeal to Theorem 1.

We form the sum $g(x) = \Sigma_{n=1}^\infty (1 - f_n(x))/2^n$, which converges for all $x \in I$, since the functions $f_n(x)$ assume only the two values ± 1. Then $\sigma(g)$ is a subfield of $\sigma(S)$, and we claim that $\sigma(g)$ is nonatomic if and only if $\Sigma\, p_n = \infty$. In fact, by a theorem of Lévy [3, pp. 16-17], the distribution function of g is continuous if and only if $\Sigma\, p_n = \infty$. In other words, $\sigma(g)$ is nonatomic if and only if $\Sigma\, p_n = \infty$. Now, if $\sigma(g)$ is nonatomic, then $\sigma(S)$ is also nonatomic, so that $\Sigma\, p_n = \infty$ implies $\sigma(S)$ is nonatomic. On the other hand, if $\sigma(S)$ is nonatomic, it is easily seen that $\sigma(g)$ is equivalent to $\sigma(S)$, so that $\Sigma\, p_n = \infty$. In summary, $\sigma(S)$ is nonatomic if and only if $\Sigma\, p_n = \infty$.

By the isomorphism theorem for measure spaces [2, p. 171], there exists a measure-preserving tranformation τ that carries $(I, \sigma(S))$ onto the Lebesgue measure space (I, β) if and only if $\sigma(S)$ is nonatomic, that is, if and only if $\Sigma\, p_n = \infty$. In other words, some member S belonging to $\{S, p_n\}$ has the property that $\sigma(S)$ is equivalent to β if and only if $\Sigma\, p_n = \infty$. An appeal to Theorem 1 completes the proof of Theorem 2.

Example. Theorem 2 does not guarantee that every member S belonging to the class $\{S, p_n\}$ has the property that S^* is complete. In fact, it is possible for S to be saturated with respect to independence and S^* to be incomplete. Consider the following example. The function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1/2, \\ 1 + \sin 2\pi(x - 1/2) & \text{if } 1/2 < x \leq 1 \end{cases}$$

generates a proper nonatomic sub-σ-field of Borel sets composed of all Borel sets of $[0, 1/2]$ and all Borel sets symmetric about 3/4 in $(1/2, 1]$. This σ-field is equivalent to the σ-field generated by the system $S = \{f_n\}_1^\infty$, where

$$f_n(x) = \begin{cases} \operatorname{sgn} \sin 2\pi\, 2^n x & \text{if } 0 \leq x \leq 1/2, \\ \operatorname{sgn} \sin 2\pi\, 2^n (x - 1/2) & \text{if } 1/2 < x \leq 1. \end{cases}$$

It is easy to verify that the members of the family S are independent, identically distributed binomial variables with $p_n \equiv 1/2$. If we assume that the generated σ-field is *not* saturated with respect to independence, then there exists a Borel set A $(0 < |A| < 1)$ independent of $\sigma(S)$. The set $C = A \cap [0, 1/2)$ belongs to $\sigma(S)$, and its measure is $|A|/2$. But $C = A \cap C$, which implies that

$$|C| = |A \cap C| = |A| \cdot |C| = |A|^2/2 < |A|/2 = |C|,$$

or $|C| < |C|$, which is a contradiction. Consequently, $\sigma(S)$ is saturated with respect to independence. On the other hand, S^* is not complete, since any function vanishing on $[0, 1/2)$ and anti-symmetric about 3/4 on $[1/2, 1]$ is orthogonal to every member of S.

3. CONDITIONAL DISTRIBUTIONS AND SATURATION WITH RESPECT TO INDEPENDENCE

In this section, it will be more convenient to discuss systems of functions, or more generally, sub-σ-fields, that are *not* saturated with respect to independence. Such sub-σ-fields will be said to admit an independent function. If a function f is independent of a subfield σ, then there exists a nontrivial set A that is independent of σ. Consequently, it is possible to say that σ is saturated with respect to independence if and only if σ does not admit an independent set.

Since the underlying space is assumed to be the Borel field of the unit interval, to each subfield σ there corresponds a family of conditional probability measures on the Borel sets, which we denote by $P_x(\cdot \,\|\, \sigma)$ for every x $(0 \leq x \leq 1)$. If σ admits a nontrivial independent Borel set A, there exists a θ $(0 < \theta < 1)$ such that $P_x(A \,\|\, \sigma) = \theta$ for almost every x in $[0, 1]$. The existence of such a value θ, simultaneously in the range of almost every conditional probability measure $P_x(\cdot \,\|\, \sigma)$, is then a *necessary condition* for σ to admit an independent set. It is possible to show that this condition is also sufficient. In fact, the existence of an independent Borel set for σ can be guaranteed under a seemingly weaker condition.

THEOREM 3. *Let* $P_x(\cdot \,\|\, \sigma)$ $(0 \leq x \leq 1)$ *be a family of conditional probability measures relative to an arbitrary subfield* σ. *Suppose that there exists a set* M $(|M| = 1)$ *and a closed interval* $0 < \theta_1 \leq \theta \leq \theta_2 < 1$ *such that for every finite set of points* $x_i \in M$ $(i = 1, \cdots, N)$ *there exist Borel sets* $\{A(x_i)\}_{i=1}^N$ *and a value* $\theta = \theta(x_1, \cdots, x_N)$ $(\theta_1 \leq \theta \leq \theta_2)$ *such that* $P_{x_i}(A_{x_i} \,\|\, \sigma) = \theta$ *for each* $i = 1, 2, \cdots, N$. *Then* σ *admits a nontrivial independent Borel set; that is, there exists a Borel set* A *such that* $P_x(A \,\|\, \sigma) = \theta$ *almost everywhere for some* θ.

Conversely, the condition is necessary for σ *to admit a nontrivial independent set.*

Proof. The necessity of the condition is clear. Now suppose that the hypothesis of Theorem 3 holds for some interval $[\theta_1, \theta_2]$ and some set M with $|M| = 1$. We make use of a theorem of Buch [1] to the effect that the range of every finite measure is a compact set. Denote by R_x the intersection of the range of $P_x(\cdot \| \sigma)$ with the closed interval $[\theta_1, \theta_2]$. The condition of Theorem 3 states that the family of compact sets R_x has the finite-intersection property, when x ranges over M. It follows that $\bigcap_x R_x \neq \emptyset$; in other words, there exists a value θ $(\theta_1 \leq \theta \leq \theta_2)$ such that to every $x \in M$, there corresponds a set A_x with the property $P_x(A_x \| \sigma) = \theta$. An independent set can be approximated in the following manner. Let $\mathscr{A} = \{A(n)\}_1^\infty$ be the collection of all finite unions of intervals with rational endpoints, and let $\{\varepsilon_k\}_1^\infty$ be a sequence of positive real numbers tending to zero. For each ε_k we construct a collection of sets $B(n, k)$ as follows:

$$B(n, k) = \{x: |P_x(A(n) \dotplus A_x \| \sigma)| \leq \varepsilon_k\},$$

where $\dotplus$ denotes symmetric difference, $B(n, k) \cap B(m, k) = \emptyset$ when $n \neq m$, and $\bigcup_{n=1}^\infty B(n, k) = M$. Now define the set $A(\varepsilon_k) = \bigcup_{n=1}^\infty (B(n, k) \cap A(n))$. Then $A(\varepsilon_k)$ is approximately independent:

$$\begin{aligned} \theta \cdot |U| - \varepsilon_k \leq |A(\varepsilon_k) \cap U| &= \int_U P_x(A(\varepsilon_k) \| \sigma)\, dx = \sum_{n=1}^\infty \int_{B(n,k) \cap U} P_x(A(n) \| \sigma)\, dx \\ &\leq \theta \cdot |U| + \varepsilon_k \end{aligned} \tag{1}$$

for each set $U \in \sigma$.

Furthermore, the sequence of sets $\{A(\varepsilon_k)\}_1^\infty$ converges to a Borel set A, since the corresponding sequence of characteristic functions is a Cauchy sequence in the L^1-norm. That is,

$$\begin{aligned} |A(\varepsilon_k) \dotplus A(\varepsilon_j)| &= \sum_{n,m} \int_{B(n,j) \cap B(m,k)} P_x(A_n \dotplus A_m \| \sigma)\, dx \\ &\leq \sum_{n,m} \int_{B(n,j) \cap B(m,k)} [P_x(A_n \dotplus A_x \| \sigma) + P_x(A_m \dotplus A_x \| \sigma)]\, dx \\ &\leq \varepsilon_k + \varepsilon_j = o(1). \end{aligned} \tag{2}$$

Inequalities (1) and (2) imply that the limiting set A is independent of the subfield σ, and the proof is complete.

Lévy [3] observed that a sufficient condition for the existence of a function independent of σ is that the conditional distribution function $F_x(y) = P_x([o, y] \| \sigma)$ be continuous for almost every x. A generalization of this follows from Theorem 3.

COROLLARY. *Let* $P_x(\cdot \| \sigma) = p(x) C_x(\cdot \| \sigma) + q(x) D_x(\cdot \| \sigma)$ *be the decomposition of* $P_x(\cdot \| \sigma)$ *into continuous and discrete components. If the discrete component*

$D_x(\cdot \,\|\, \sigma)$ *with weight* $q(x)$ *is such that* $\operatorname{ess\,sup}_{0 \leq x \leq 1} q(x) < 1$, *then* σ *admits an independent set.*

Proof. Since $q(x) + p(x) = 1$, the condition guarantees that $p(x) \geq \theta > 0$ for almost every x; that is, the range of almost every $P_x(\cdot \,\|\, \sigma)$ contains the value θ, so that by Theorem 3 an independent set exists.

REFERENCES

1. K. R. Buch, *Some investigations of the set of values of measures in abstract space*, Danske Vid. Selsk. Math.-Fys. Medd. 21 (1945), no. 9, 70 pp.

2. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, New Jersey, 1950.

3. P. Lévy, *Théorie de l'addition des variables aléatoires*, Second Edition, Gauthier-Villars, Paris, 1954.

4. A. Rényi, *On a conjecture of H. Steinhaus*, Ann. Soc. Polon. Math. 25 (1952), 279-87.

5. A. Wintner, *Asymptotic distributions and infinite convolutions*, Edwards Bros., Ann Arbor, Michigan, 1938.

The University of Michigan
Ann Arbor, Michigan