# THE DEGREE OF APPROXIMATION BY POSITIVE CONVOLUTION OPERATORS

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1. *Introduction.* Let C(T) denote the space of continuous functions defined on the unit circle T. If $p_n(t)$ is a trigonometric polynomial of degree at most n and f is a nonconstant function in C(T), it is clear that as $n \to \infty$,

$$(L_n f)(x) = (f * p_n)(x) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) p_n(x - t)\, dt$$

cannot converge faster to f(x) in the sup norm than the Fourier coefficients of $p_n$ approach 1. Thus it follows that if $L_n f$ is the arithmetic mean $\sigma_n f$ of the partial sums of the Fourier series for f, and if $\|f - \sigma_n f\| = o(1/n)$, then f must be identically constant, a fact first observed by Hille [3].

It is the purpose of this paper to show that if for each n, $p_n(t)$ is a nonnegative trigonometric polynomial of degree at most n and $L_n f = f * p_n$, then

$$\|L_n f - f\| = o\left(\frac{1}{n^2}\right)$$

implies that f is identically constant, provided

$$\hat{p}_n(0) - 1 = o\left(\frac{1}{n^2}\right), \qquad \text{where } \hat{p}_n(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_n(t) e^{-kit}\, dt.$$

There exist well-known examples of positive convolution operators of this type, with the property that for certain nonconstant functions the convergence is $O(n^{-2})$. Our theorem extends an earlier result of Korovkin [5], who showed that in general one can not achieve the best polynomial approximation to f(t) by a sequence of positive operators, if that approximation is $o(n^{-2})$.

Theorems of this type are known for certain choices of the operators $L_n$ (see for example Butzer [2]). A somewhat similar result for approximation in the sup norm to continuous functions on the unit interval by Bernstein polynomials has been proved by de Leeuw [4]. He showed that $\|B_n f - f\| = o(1/n)$ implies that f is linear.

Since the main results are valid also in the spaces $L_p(T)$ $(1 \leq p < \infty)$, we adopt the following terminology. We denote by E any one of the Banach spaces C(T) and $L_p(T)$ $(1 \leq p < \infty)$. For $f \in E$, and a nonnegative trigonometric polynomial $p_n$, the formula

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$$(L_n f)(x) = (f * p_n)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,p_n(x-t)\,dt$$

defines a bounded linear operator in E that commutes with translation by points in T. Translation by $x \in T$ will be written $U_x$, so that $(U_x f)(t) = f(x+t)$. The symbol $L_n$ will always indicate an operator of the above type.

In Section 3 we prove that if $f \in E$ and $\|L_n f - f\| = o(n^{-2})$, then f is constant provided that $\hat{p}_n(0) - 1 = o(n^{-2})$. We also give a generalization, in the case that $E = C(T)$, to operators that do not commute with translation. In Section 2 we discuss results of Korovkin that give conditions on a sequence of positive linear operators $T_n$ in $C(T)$ guaranteeing that

$$\|T_n f - f\| \to 0 \quad \text{for each } f \text{ in } C(T).$$

These results extend to the spaces $L_p$, with little or no change in proof. Specifically, if $\{T_n\}$ is a sequence of positive linear operators in E ($f \geq 0$ almost everywhere implies $T_n f \geq 0$ almost everywhere), then $\|T_n f - f\| \to 0$ for each f, if

$$\sup_n \|T_n\| < \infty \quad \text{and} \quad \|T_n e_k - e_k\| \to 0 \quad \text{for } k = 0, 1,$$

where $e_k(x) = e^{kix}$. This should be contrasted with standard results on uniform boundedness requiring that $\|T_n e_k - e_k\| \to 0$ for all k.

2. We now prove the theorem of Korovkin for the space E.

2.1 THEOREM. *Let* $T_n$ *be a uniformly bounded sequence of positive linear operators in* E. *Then* $\lim_{n\to\infty} T_n f = f$ *for each* $f \in E$ *provided that* $\lim_{n\to\infty} T_n e_k = e_k$ *for* $k = 0, 1$.

*Proof.* Since the continuous functions are dense in E, $T_n f \to f$ for each $f \in E$ provided this is true for each continuous f. Positivity of the operator $T_n$ implies that for f real and continuous, $T_n f$ is real. Moreover, positivity coupled with the assumption that $T_n e_k \to e_k$ for $k = 0, 1$ implies that $T_n e_{-1} \to e_{-1}$.

So we assume that f is real and continuous, and we let $\psi(t) = \sin^2 t/2$. For $\varepsilon > 0$, pick $\delta$ such that $|t - x| < \delta$ implies $|f(t) - f(x)| < \varepsilon$. Then, if $M = \|f\|_\infty$, it follows that for all t and $x \in T$

$$|f(t) - f(x)| \leq \varepsilon + \frac{2M}{\sin^2 \delta/2}\,\psi(t-x).$$

For $|t - x| < \delta$ this is clear; and if $|t - x| \geq \delta$, then $\psi(t-x)/(\sin^2 \delta/2) \geq 1$. Applying the positivity of $T_n$, we have for each $x \in T$ and all $t \notin F_x$ ($F_x$ a null set of T depending on x) the inequality

$$|(T_n f)(t) - f(x)(T_n e_0)(t)| \leq \varepsilon (T_n e_0)(t) + \frac{2M}{\sin^2 \delta/2}\,(T_n U_{-x}\psi)(t).$$

From this it follows that for almost all x

$$|(T_n f)(x) - f(x)(T_n e_0)(x)| \leq \varepsilon (T_n e_0)(x) + \frac{2M}{\sin^2 \delta/2}\,(T_n U_{-x}\psi)(x).$$

Here we use the fact that if the $g_i$ are measurable on T, the $h_i$ are continuous, and $\Sigma_{i=1}^n g_i(t)h_i(x) \geq 0$ for each x and all $t \notin F_x$ ($F_x$ a null set depending on x), then

$$\left\{(t, x): \sum_{i=1}^{n} g_i(t)h_i(x) < 0\right\}$$

has zero product measure. From this it follows that $\Sigma_{i=1}^n g_i(x)h_i(x) \geq 0$ almost everywhere. For if $\Sigma_{i=1}^n g_i(x)h_i(x) < 0$ on a set E of positive measure, then by the continuity of $h_i$, for each $x_0 \in E$ there exists an open set U such that $x_0 \in U$ and

$$\sum_{i=1}^{n} g_i(x_0)h_i(x) < 0 \qquad (x \in U).$$

This implies $\{(t, x): \Sigma_{i=1}^n g_i(t)h_i(x) < 0\}$ has positive measure, which is a contradiction. Therefore, letting $\phi_n(x) = (T_n U_{-x}\psi)(x)$, we have the inequality

$$\|T_n f - f\| \leq \varepsilon \|T_n e_0\| + \frac{2M}{\sin^2 \delta/2} \|\phi_n\| + \|f\|_\infty \|1 - T_n e_0\|.$$

To complete the proof, we need only show that $\|\phi_n\| \to 0$. Since

$$\phi_n(x) = \frac{(T_n e_0)(x)}{2} - \frac{1}{4}\left[\frac{(T_n e_1)(x)}{e_1(x)} + \frac{(T_n e_{-1})(x)}{e_{-1}(x)}\right],$$

this is clear.

2.2 COROLLARY. *Let* M(T) *denote the space of regular Borel measures on* T. *Let* $\{\mu_n\} \subset M(T)$, *and assume* $\mu_n \geq 0$ *for each* n. *Then* $\lim_{n\to\infty} \mu_n^* f = f$ *for each* $f \in E$ *provided that* $\lim_{n\to\infty} \hat{\mu}_n(k) = 1$ *for* $k = 0, 1$.

*Proof.* $\|\mu_n^* f\| \leq \|f\| \int_{-\pi}^{\pi} d\mu_n(t) = 2\pi |\hat{\mu}_n(0)| \|f\|$. Therefore the operators $T_n$ defined by $T_n f = \mu_n^* f$ are uniformly bounded in norm. Since, for each n, $T_n$ commutes with translation, the requirement that $\lim_{n\to\infty} T_n e_k = e_k$ for $k = 0, 1$ reduces to the statement that

$$\lim_{n\to\infty} \hat{\mu}_n(k) = 1 \qquad \text{for } k = 0, 1.$$

3. We now discuss the degree of approximation afforded by positive operators in E that are convolutions by nonnegative trigonometric polynomials.

3.1 THEOREM. *Let* $L_n f = f * p_n$, *where* $p_n$ *is a nonnegative trigonometric polynomial of degree at most* n. *Assume* $1 - \hat{p}_n(0) = o(n^{-2})$. *Then if* $f \in E$ *and* $\|L_n f - f\| = o(n^{-2})$, *it follows that* f *is identically constant.*

The proof proceeds by a number of lemmas. First we need an extension of a well-known result concerning the degree of approximation in the sup norm to $|\sin x|$ by trigonometric polynomials of degree $\leq n$. Denoting this class of polynomials by $\pi_n$, we define for $f \in C(T)$

$$E_n(f) = \inf_{g \in \pi_n} \|f - g\|_\infty.$$

3.2 LEMMA. *For each positive integer* k, $\limsup_{n \to \infty} E_n(|\sin kx|) \geq k/4\pi$

*Proof.* Let $\delta_n = E_n(|\sin x|)$. As is well known [5, p. 165], $\delta_n \geq 1/4\pi n$. Let $q_n$ be the unique trigonometric polynomial in $\pi_n$ that approximates $|\sin x|$ best in the sup norm. The uniqueness of $q_n$ guarantees that $q_n$ is even and $q_n(0) = q_n(\pi)$. By the fundamental theorem of de La Vallée Poussin and Bernstein concerning best approximation by Tchebycheff families [1, p. 57], $q_n$ is uniquely characterized by the existence of at least $2n + 2$ distinct points $x_i \in T$ such that if $x_1 < x_2 < \cdots < x_{2n+2}$, then either

$$|\sin x_i| - q_n(x_i) = (-1)^i \, \| \, |\sin(\cdot)| - q_n \|_\infty$$

or

$$|\sin x_i| - q_n(x_i) = (-1)^{i+1} \, \| \, |\sin(\cdot)| - q_n \|_\infty .$$

Therefore, if $m = kn$, $|\sin kx| - q_n(kx)$ alternates with equal magnitude at at least $k(2n + 2) \geq 2kn + 2$ points $x_i \in T$. Since $q_n(kx) \in \pi_n$, we see that $q_n(kx) = q_m(x)$. Consequently

$$E_m(|\sin kx|) = E_n(|\sin x|) \geq \frac{1}{4\pi n} = \frac{k}{4\pi m},$$

which proves the lemma.

Next we need an extension of a result of Korovkin [5, p. 128]. His result is the case $k = 1$. The proof proceeds similarly.

3.3 LEMMA. *For each positive integer* n, *let* $T_n$ *be a positive linear operator mapping* C(T) *into* $\pi_n$, *and suppose that* $\|T_n e_0 - e_0\|_\infty = o(n^{-1})$. *Also, let*

$$f_k(t) = \left|\sin \frac{kt}{2}\right| \quad \textit{and} \quad g_{n,k}(x) = (T_n U_{-x} f_k^2)(x).$$

*Then for each integer* $k \neq 0$, $\|g_{n,k}\|_\infty \neq o(n^{-2})$.

*Proof.* The identity $\sin a - \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}$ shows that for all t, $x \in T$,

$$\big| |\sin kt| - |\sin kx| \big| \leq |\sin kt - \sin kx|$$

$$= 2 \left|\cos \frac{k}{2}(t + x) \sin \frac{k}{2}(t - x)\right| \leq 2 \left|\sin \frac{k}{2}(t - x)\right| = 2(U_{-x} f_k)(t).$$

Therefore, letting $h_k(t) = |\sin kt|$ and using the positivity of $T_n$, we have the inequality

$$|(T_n h_k)(t) - |\sin kx| (T_n e_0)(t)| \leq 2(T_n U_{-x} f_k)(t),$$

and for $t = x$,

$$\big| (T_n h_k)(x) - |\sin kx| \big| \leq |\sin kx| \, |1 - (T_n e_0)(x)| + 2(T_n U_{-x} f_k)(x)$$

$$\leq |\sin kx| \, |1 - (T_n e_0)(x)| + 2[(T_n U_{-x} f_k^2)(x) (T_n e_0)(x)]^{1/2} .$$

The last inequality follows from the Cauchy-Schwartz inequality, valid for positive functionals on C(T). Using the fact that $T_n h_k \in \pi_n$, we see that

$$E_n(|\sin kx|) \leq \|T_n h_k - |\sin k(\cdot)|\|_\infty \leq \|e_0 - T_n e_0\|_\infty + \text{const.}\ \|g_{n,k}\|_\infty^{1/2}.$$

By Lemma 3.2 it is impossible that $\|g_{n,k}\|_\infty = o(n^{-2})$.

3.4 LEMMA. *Let* $p_n \in \pi_n$ *and assume* $p_n \geq 0$. *If* $1 - \hat{p}_n(0) = o(n^{-1})$, *then for each positive integer* k

$$\hat{p}_n(0) - \frac{\hat{p}_n(k) + \hat{p}_n(-k)}{2} \neq o(n^{-2}).$$

*Proof.* Define $L_n f = f * p_n$. Then

$$(L_n U_{-x} f_k^2)(x) = (L_n f_k^2)(0) = \frac{1}{2}\hat{p}_n(0) - \frac{1}{4}[\hat{p}_n(k) + \hat{p}_n(-k)],$$

and the result follows from Lemma 3.3.

We now prove Theorem 3.1. Suppose for some $f \in E$, $\|L_n f - f\| = o(n^{-2})$. We assert that $\hat{f}(k) = 0$ for $k \neq 0$. If $\hat{f}(k) \neq 0$ for some $k \neq 0$, then

$$|(\hat{p}_n(k) - 1)\hat{f}(k)| \leq \|L_n f - f\|.$$

Hence $(\hat{p}_n(k) - 1) = o(n^{-2})$, and by taking real and imaginary parts we see that $\hat{p}_n(-k) - 1 = o(n^{-2})$. Therefore

$$1 - \frac{\hat{p}_n(k) + \hat{p}_n(-k)}{2} = o(n^{-2}),$$

and since $1 - \hat{p}_n(0) = o(n^{-2})$, we have a contradiction of Lemma 3.4.

3.5 COROLLARY. *The conclusions of Theorem* 3.1 *hold if in place of the assumption that* $1 - \hat{p}_n(0) = o(n^{-2})$ *we assume* $\hat{p}_n(0) \leq 1$.

*Proof.* $\hat{p}_n(0) \geq |\hat{p}_n(k)|$. Therefore $1 - \hat{p}_n(k) = o(n^{-2})$ for some $k \neq 0$ implies $1 - \hat{p}_n(0) = o(n^{-2})$.

For general positive operators mapping E into $\pi_n$, the problem of the rate of convergence seems to be much more complicated. For E = C(T), we can give an estimate on the best rate of convergence for a certain class of smoothed interpolation operators. Specifically, we consider operators of the following type. Suppose $p_n \in \pi_n$ and $p_n \geq 0$. Define

$$(P_n f)(x) = \frac{1}{2\pi m_n} \sum_{k=0}^{m_n - 1} f\left(-\pi + \frac{2k\pi}{m_n}\right) p_n\left(x + \pi - \frac{2k\pi}{m_n}\right),$$

which we write $\int_{-\pi}^{\pi} f(t) p_n(x - t)\, d\mu_n(t)$. $\mu_n$ is a discrete measure, with total mass $2\pi$, and with equal point masses at intervals of length $2\pi/m_n$.

3.6 THEOREM. *If* $1/m_n = o(n^{-1})$, $\hat{p}_n(0) - 1 = o(n^{-2})$, *and* $\|f - P_n f\|_\infty = o(n^{-2-\alpha})$ *for some* $\alpha > 0$, *then* f *is constant.*

*Proof.* By the familiar theorem of Bernstein [5, p. 95], the hypothesis implies that $f^{(2)}$ is continuous. For any such function,

$$(*) \qquad \left| \int_{-\pi}^{\pi} f(x)\,dx - \int_{-\pi}^{\pi} f(x)\,d\mu_n(x) \right| = o(m_n^{-2}),$$

by the error estimate for the trapezoidal formula. Therefore, letting

$$\tilde{f}_n(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,e^{-kit}\,d\mu_n(t),$$

we have the relations

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-kit}\,[f - P_n f](t)\,dt = \hat{f}(k) - \hat{p}_n(k)\tilde{f}_n(k)$$

$$= \hat{f}(k)\,[1\,\hat{p}_n(k)] + \hat{p}_n(k)\,[\hat{f}(k) - \tilde{f}_n(k)].$$

By (*), $\hat{f}(k) - \tilde{f}_n(k) = o(m_n^{-2})$. Therefore $\|f - P_n f\| = o(n^{-2})$ implies

$$1 - \hat{p}_n(k) = o(n^{-2}),$$

which is a contradiction unless $\hat{f}(k) = 0$.

It seems doubtful that this estimate is best possible.

## REFERENCES

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