ASYMPTOTIC VALUES OF FUNCTIONS HOLOMORPHIC IN THE UNIT DISC

J. E. McMillan

1. INTRODUCTION

This paper was written under the direction of G. R. MacLane, and it is the author's Ph.D. thesis at Rice University. Its main result is an extension of theorems of Bagemihl and Seidel [3, Theorem 3] and MacLane [4, Theorem 11] on the asymptotic values of a function f holomorphic in the unit disc. We say that f *has the asymptotic value* a *at* ζ ($|\zeta| = 1$) *if there exists a Jordan arc that lies in* $\{|z| < 1\}$, *except for the endpoint* ζ, *and on which* f *has the limit* a *at* ζ.

MacLane [4] considered the class $\mathscr{A}$ of nonconstant holomorphic functions having asymptotic values at a dense set of points on $\{|z| = 1\}$. In particular, he proved that *if* $f \in \mathscr{A}$ *and* γ *is an arc of* $\{|z| = 1\}$, *then either* f *has the asymptotic value* ∞ *at a point of* γ *or* f *has point asymptotic values at points of a subset of* γ *of positive Lebesgue measure.* We shall prove a global version of this theorem without the hypothesis $f \in \mathscr{A}$. As corollaries we find that f either has the asymptotic value ∞ or has point asymptotic values on a set of positive measure, and that an f with only finitely many tracts for ∞ must either have only finitely many tracts or have asymptotic values at points of a set of positive measure (for the definition of the concept of a tract, see Section 2). Several related results are also obtained.

2. DEFINITIONS

The following notation will be used throughout this paper. Let $D = \{|z| < 1\}$ and $C = \{|z| = 1\}$. Let f be a function holomorphic in D. For any subset S of the sphere, let

$$A(S) = \{\zeta \in C: \text{there exists } a \in S \text{ such that f has the asymptotic value } a \text{ at } \zeta.\}$$

In particular, we let

$$A = A \text{ (the sphere)}, \qquad A_\infty = A(\{\infty\}), \qquad A^* = A - A_\infty.$$

The Lebesgue measure and exterior Lebesgue measure (in $[0, 2\pi]$) of a subset B of C will be denoted by m(B) and $m_e(B)$. The interior of an arc $\gamma \subset C$ will be denoted by γ^0. If Δ is a plane domain, $\partial\Delta$ will denote the boundary of Δ. The closure of a set S in the plane will be denoted by $\overline{S}$. Also, we write

$$\{|f| > \lambda\} = \{z: |f(z)| > \lambda\}.$$

Let a be a complex number, and suppose that for each $\varepsilon > 0$, $D(\varepsilon)$ is a component of $\{z: |f(z) - a| < \varepsilon\}$; suppose further that $D(\varepsilon_1) \subset D(\varepsilon_2)$ ($\varepsilon_1 < \varepsilon_2$) and

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$\bigcap_{\varepsilon>0} D(\varepsilon) = \square$. Then the family $\{D(\varepsilon)\}$ is a *tract* (or *asymptotic tract*) of f for the value a. We have the analogous definition of tract of f for the value ∞. The set $K = \bigcap_{\varepsilon>0} \overline{D(\varepsilon)}$ is called the *end* of the tract $\{D(\varepsilon)\}$. The tract is called a *point tract* or an *arc tract* depending on whether K is a point or an arc of C. We say that the curve described by $z(t)$ $(0 \leq t < 1)$ *belongs to the tract* $\{D(\varepsilon)\}$ if to each $\varepsilon > 0$ there corresponds a $t(\varepsilon) < 1$ such that $z(t) \in D(\varepsilon)$ whenever $t > t(\varepsilon)$. For a discussion of tracts, see MacLane [4].

If Γ is the curve in D described by $z(t)$ $(0 \leq t < 1)$ and $|z(t)| \to 1$ as $t \to 1$, then we write $\Gamma \to S$ if and only if $S = \overline{\Gamma} \cap C$. If $\Gamma \cup \{\zeta\}$ is a Jordan arc $(|\zeta| = 1)$, then we say that Γ is an arc in D tending to ζ. If $\{\gamma_n\}$ is a sequence of curves in D and γ is the arc $\{e^{i\theta}: \theta_1 \leq \theta \leq \theta_2\}$ $(0 \leq \theta_1 \leq \theta_2 < 2\pi)$ of C, then we write $\gamma_n \to \gamma$ if and only if to each $\varepsilon > 0$ there corresponds an $n(\varepsilon)$ such that

$$\gamma_n \subset \{z: 1 - \varepsilon < |z| < 1, \ \theta_1 - \varepsilon < \arg z < \theta_2 + \varepsilon\}$$

and

$$\gamma_n \cap \{z: 1 - \varepsilon < |z| < 1, \ \theta_i - \varepsilon < \arg z < \theta_i + \varepsilon\} \neq \square \qquad (i = 1, 2)$$

whenever $n \geq n(\varepsilon)$. For the special case where $\theta_1 = \theta_2$, this defines the meaning of the statement $\gamma_n \to e^{i\theta_1}$.

3. A PRELIMINARY THEOREM

In this section we extend (by generalizing MacLane's proof) the theorem of MacLane [4, Theorem 10] for $f \in \mathscr{A}$. The measurability of A^* is needed in the proof of Theorem 2.

THEOREM 1. *If* S *is a Borel set on the sphere, then* A(S) *is a Borel set.*

Proof. As in the proof of MacLane [4, Theorem 10], it suffices to consider the two cases where S is closed and bounded and where $S = \{\infty\}$. Suppose first that S is closed and bounded. For each $n \geq 1$, let $\Delta(n, 1), \cdots, \Delta(n, \nu_n)$ be a finite set of open discs of radius 4^{-n} that covers S and is not redundant, in the sense that

$$\Delta(n, k) \cap S \neq \square \qquad (1 \leq k \leq \nu_n). \tag{3.1}$$

Let $\Delta^*(n, k)$ be the open disc with radius $2 \cdot 4^{-n}$ and the same center as $\Delta(n, k)$. We suppose that the discs Δ and Δ^* have been chosen so that their circumferences contain no projections of the branch points of the Riemann surface $\mathscr{S}$ onto which f maps D. Let $D(n, k, p)$ $(p \geq 1)$ be the domains in D that correspond to the components of $\mathscr{S}$ over $\Delta(n, k)$. Then each $D(n, k, p)$ is bounded by level curves, without multiple points, of $f - a(n, k)$, where $a(n, k)$ is the center of $\Delta(n, k)$, and possibly also by a subset of C. Let $E(n, k, p) = C \cap \partial D(n, k, p)$.

Now the set

$$U = \{\zeta \in C: \text{there exist numbers } a, \lambda, \text{ and arcs } \gamma_i \subset \{|f - a| = \lambda\} \text{ and } \gamma \subset C \text{ such that } \gamma_i \to \gamma \text{ and } \zeta \in \gamma^0\} \tag{3.2}$$

is open. Let P_1 denote the countable set of endpoints of the components of U. Let P_2 denote the countable set of points $\zeta \in C$ such that for some (n, k, p) there is a level curve on the boundary of D(n, k, p) that tends to the point ζ. Let

(3.3) $$E'(n, k, p) = E(n, k, p) - U$$

and

(3.4) $$H(n, k, p) = E'(n, k, p) - (P_1 \cup P_2) = E(n, k, p) - (U \cup P_1 \cup P_2).$$

Then all the sets

$$E'(n) = \bigcup_{(k,p)} E'(n, k, p), \qquad H(n) = \bigcup_{(k,p)} H(n, k, p),$$

(3.5) $$E' = \bigcap_{n=1}^{\infty} E'(n), \qquad H = \bigcap_{n=1}^{\infty} H(n)$$

are Borel sets. Also,

(3.6) $$E' = H \cup \{\text{countable set}\}.$$

LEMMA 1. *The set* $U \cap A(S)$ *is countable.*

Proof. Since U is covered by the arcs γ^0 in (3.2), it is covered by a countable subcollection of these arcs. Thus it suffices to show that if γ is one such arc, then $\gamma^0 \cap A(S)$ contains at most one point. Suppose this were not the case. Choose $\zeta_1, \zeta_2 \in \gamma^0$ and arcs $\Gamma_i \to \zeta_i$ such that $f \to a_i$ on Γ_i $(i = 1, 2)$. We may suppose that each γ_n (see (3.2)) crosses both Γ_1 and Γ_2. Now choose $N > 0$ sufficiently large, and let Δ be the component of $\{|f| < N\}$ containing $\left(\bigcup \gamma_i\right) \cup \Gamma_1 \cup \Gamma_2$. Since Δ is simply connected, it must contain a neighborhood of some interior point of γ. This contradicts a lemma of MacLane [4, Lemma 1] to the effect that level sets of bounded functions end at points (see [4, p. 8]). Thus Lemma 1 is proved.

Finally, let

(3.7) $$A'(S) = A(S) - U.$$

We shall prove that

(3.8) $$H \subset A'(S) \subset E'.$$

It will then follow from (3.6), (3.7), (3.8), and Lemma 1 that

(3.9) $$A(S) = H \cup \{\text{countable set}\}.$$

Thus, since H is a Borel set, Theorem 1 will have been proved in the first case.

We show first that $A'(S) \subset E'$. If $\zeta \in A'(S)$, then $f \to a \in S$ on a curve $\Gamma \to \zeta$, and $\zeta \notin U$. For each n, $a \in \Delta(n, k_n)$ for some k_n, the curve Γ lies eventually in some $D(n, k_n, p_n)$, and therefore $\zeta \in E(n, k_n, p_n)$. But $\zeta \notin U$, so that $\zeta \in E'(n, k_n, p_n)$ and $\zeta \in E'(n)$ for each n. Thus $\zeta \in E'$.

Now choose $\zeta \in H$. For each n there is a pair (k_n, p_n) such that

$$\zeta \in H(n, k_n, p_n).$$

We shall prove that any two of the corresponding sets $D(n, k, p)$ must intersect. Let

$$D_1 = D(n_1, k_{n_1}, p_{n_1}), \qquad D_2 = D(n_2, k_{n_2}, p_{n_2}),$$

and suppose $D_1 \cap D_2 = \Box$. It is easy to see that we may let Γ_i be a curve (not necessarily simple) in D_i tending to ζ, or an arc γ_i containing ζ $(i = 1, 2)$. Let Γ be a Jordan arc in D joining the loose ends of the Γ_i. Then, since only finitely many components of ∂D_1 can meet Γ, it is clear that one of these components must separate Γ_1 and Γ_2 and tend either to an arc γ containing ζ or else to ζ. In the first case, $\zeta \in U \cup P_1$, and in the second case, $\zeta \in P_2$. In either case we have a contradiction from (3.4). Thus $D_1 \cap D_2 \neq \Box$ and

$$\Delta(n_1, k_{n_1}) \cap \Delta(n_2, k_{n_2}) \neq \Box.$$

Thus, from the choice of the radii of the discs $\Delta(n, k)$ and $\Delta^*(n, k)$, we see that

$$\Delta^*(n+1, k_{n+1}) \subset \Delta^*(n, k_n) \qquad (n \geq 1). \tag{3.10}$$

Since the diameters of $\Delta^*(n, k_n)$ tend to zero, it follows from (3.1) that

$$\bigcap \Delta^*(n, k_n) = \{a\} \qquad (a \in S). \tag{3.11}$$

Now let D_n^* be the component of $\{z: f(z) \in \Delta^*(n, k_n)\}$ containing $D(n, k_n, p_n)$. Then from (3.10) we have the relation

$$D_{n+1}^* \subset D_n^* \qquad (n \geq 1). \tag{3.12}$$

Let Γ^* be a curve that lies eventually in each D_n^* and tends to an arc γ^* containing ζ, or to ζ. If $\Gamma^* \to \zeta$, the relation $\zeta \in A'(S)$ follows from (3.4) and (3.11). Suppose then that $\Gamma^* \to \gamma^*$. We shall prove that $\gamma^* \subset U \cup P_1$, and contradict the relation $\zeta \in H$. It suffices to show that at most one interior point of γ^* is not contained in U (we suppose $\gamma^* \neq C$; modifications for the case $\gamma^* = C$ are obvious).

Let ζ_1 and ζ_2 be two distinct interior points of γ^*. Choose ζ_3 in the open subarc of γ^* determined by ζ_1 and ζ_2. For each $\varepsilon > 0$, let

$$N_\varepsilon = D \cap \{|z - \zeta_3| < \varepsilon\}.$$

We suppose that ε is such that $\zeta_1, \zeta_2 \notin \{|z - \zeta_3| < \varepsilon\}$. Now suppose that $N_\varepsilon \cap (\partial D_1^*) = \Box$ for some ε. Then $N_\varepsilon \subset D_1^*$, and f is bounded in N_ε. But we can choose a sequence of arcs $\gamma_n' \subset \Gamma^* \cap N_\varepsilon$ such that

$$\gamma_n' \to (\partial N_\varepsilon) \cap C \qquad \text{and} \qquad \max_{z \in \gamma_n'} |f(z) - a| \to 0 \quad (n \to \infty)$$

(see (3.11)). This contradicts Koebe's lemma. Thus, for sufficiently small ε,

$$N_\varepsilon \cap (\partial D_1^*) \neq \Box\,. \tag{3.13}$$

Now let K be the (open) sector bounded by the radii R_1 and R_2 and the (closed) arc γ' contained in the interior of γ^* and containing ζ_1 and ζ_2 in its interior. We may choose Jordan arcs $\gamma_n \subset \Gamma^*$ such that $\gamma_n^0 \subset K$, γ_{n+1} separates γ_n from γ', and $\gamma_n \to \gamma'$. Let K_n be the subdomain of K lying between γ_n and γ_{n+1}. Choose $\varepsilon_n \downarrow 0$. Using (3.13), we choose

$$z_n \in N_{\varepsilon_n} \cap (\partial D_1^*).$$

By extracting subsequences of $\{z_n\}$ and $\{\gamma_n\}$ if necessary, we may suppose $z_n \in K_n$ $(n \geq 1)$. Now z_n is in some component, say C_n, of ∂D_1^*, and since $a \in \Delta_1^*$, for sufficiently large n $(n \geq n_0)$, C_n cannot intersect γ_n or γ_{n+1}. Thus, since D_1^* is simply connected,

$$C_n \cap (R_1 \cup R_2) \neq \square \qquad (n \geq n_0).$$

It follows clearly that some subsequence of $\{C_n\}$ tends to an arc containing ζ_1 or ζ_2. Therefore either ζ_1 or ζ_2 is in U, hence $H \subset A'(S)$, and (3.8) is proved. Thus Theorem 1 has been proved in the case where S is closed and bounded.

Suppose now that $S = \{\infty\}$. Let $\Delta_n = \{|w| > n\}$, and suppose that the level sets $\{|f| = n\}$ have no double points. Let $D(n, p)$ $(p \geq 1)$ be the components of $\{z: f(z) \in \Delta_n\}$. Let $E(n, p) = C \cap (\partial D(n, p))$, and set

$$E'(n, p) = E(n, p) - U,$$

$$H(n, p) = E(n, p) - (U \cup P_1 \cup P_2).$$

Then, as before, the sets

$$E'(n) = \bigcup_{p=1}^{\infty} E'(n, p) \quad \text{and} \quad H(n) = \bigcup_{p=1}^{\infty} H(n, p),$$

together with the sets E' and H defined by (3.5), are Borel sets. We shall prove that

$$H \subset A(S) \subset E'. \tag{3.14}$$

Then (3.9) and the conclusion of the theorem will follow as before.

The inclusion $A(S) \subset E'$ follows easily, as before. We now prove that $H \subset A(S)$. Choose $\zeta \in H$. For each n there is a p_n such that $\zeta \in H(n, p_n)$. Let $D_n = D(n, p_n)$. Then with only minor modifications in the previous argument we see that $D_{n+1} \subset D_n$ $(n \geq 1)$. Now let Γ^* be a curve that is eventually in each D_n and tends to an arc γ^* containing ζ, or to ζ. If $\Gamma^* \to \zeta$, then $\zeta \in A(S)$. Suppose then that $\Gamma^* \to \gamma^*$. Using an argument of MacLane [4, Theorem 3], we construct a curve tending to ζ on which f tends to ∞. Let $\{\gamma_n\}$ be a sequence of pairwise disjoint Jordan arcs contained in Γ^* such that $\gamma_n \to \gamma^*$ (we suppose for simplicity that $\gamma^* \neq C$). Then

$$\inf_{z \in \gamma_n} |f(z)| \to \infty \quad \text{as } n \to \infty.$$

We suppose first that $\zeta \in \gamma^{*0}$. Let $R = \{z \in D: \arg z = \arg \zeta\}$ be the radius at ζ. Since $\zeta \notin U \cup P_1$, it follows that to each positive integer n there corresponds an $\varepsilon(n) > 0$ such that each component of $\{|f| = n\}$ meeting $R \cap \{|z - \zeta| < \varepsilon(n)\}$ is compact. Thus, again since $\zeta \notin U \cup P_1$, there exists a curve Γ_n tending to ζ such that $\Gamma_n \subset \{|f| \geq n\}$. We can choose the curves Γ_n so that any two of them intersect arbitrarily near ζ. It follows that there exists a curve tending to ζ on which f tends to ∞. Suppose now that ζ is an endpoint of γ^*. Since $\zeta \notin U \cup P_1$, there exists a sequence

$$\{\zeta_n\} \subset \gamma^{*0} \cap [C - (U \cup P_1)]$$

such that $\zeta_n \to \zeta$. Then there are curves $\Gamma_n \to \zeta_n$ on which $f \to \infty$. Since each Γ_n meets all but finitely many of the arcs γ_m, it follows that there exists a curve tending to ζ on which $f \to \infty$. Thus the proof of Theorem 1 is complete.

4. THE MAIN THEOREM

We now state the main theorem of this paper.

THEOREM 2. *If*

(4.1) $\quad$ f *has only finitely many tracts for* ∞

and

(4.2) $\quad$ *the ends of the arc tracts of* f *for* ∞ *do not cover* C,

then $m(A^*) > 0$.

Note that (4.1) and (4.2) imply the existence of an arc of C that meets the end of no tract of f for the value ∞.

Remarks. The modular function, which maps D onto the universal covering surface of the sphere punctured at the three points 0, 1, and ∞ satisfies (4.2), but not (4.1), and it has point asymptotic values at only countably many points of C. Example 1 [see Section 5 for examples], which has only one asymptotic tract, satisfies (4.1), but not (4.2), and does not satisfy the conclusion of the theorem. Example 4 shows that a local version of Theorem 2 cannot be true. That is, if γ is any closed arc of C not containing the point 1, then the end of each asymptotic tract of f for the value ∞ has void intersection with γ, and yet $A \cap \gamma = \square$. Example 5 shows that to each $\varepsilon > 0$ there corresponds an f satisfying (4.1) and (4.2) for which A^* is contained in an arc of length ε.

Theorem 2 will follow simply from the following lemma.

LEMMA 2. *If* f *has only finitely many distinct point tracts for* ∞ *ending at any one point of* C *and if there exists a* $\lambda > 0$ *such that* f *is bounded in some component of* $\{|f| > \lambda\}$, *then* $m(A^*) > 0$.

Proof. It suffices to consider the case

$$\limsup_{z \to \zeta} |f(z)| = \infty \qquad (\text{each } \zeta \in C). \tag{4.3}$$

Let D^* be a component of $\{|f| > \lambda\}$ in which f is bounded. We may suppose that the level set $\{|f| = \lambda\}$ has no double points. Now add to D^* the interiors of all Jordan curves in D^*, and let D_0 be the resulting simply connected domain. Let

$$\Gamma = (\partial D_0) \cap D \qquad \text{and} \qquad F = (\partial D_0) \cap C.$$

Then $|f(z)| = \lambda$ on Γ and f is bounded in D_0. We suppose $0 \in D_0$.

Let $w(z)$ be a one-to-one conformal map of D_0 onto $\{|w| < 1\}$ with $w(0) = 0$, and let $z(w)$ be the inverse map. Then $z(w)$ and $f(z(w))$ are both bounded in $\{|w| < 1\}$, and by Fatou's theorem they have radial limits almost everywhere. Let

(4.4) $$\Phi_1 = \{\phi: z(e^{i\phi}) \text{ and } f(z(e^{i\phi})) \text{ both exist}\}.$$

Then $m(\Phi_1) = 2\pi$. Since $f(z(w))$ has a Poisson integral representation in terms of $f(z(e^{i\phi}))$ and $|f(z)| > \lambda$ in D^*, we know that $|f(z(e^{i\phi}))| > \lambda$ for ϕ in a set Φ_2 of positive measure. Let

(4.5) $$\Phi = \Phi_1 \cap \Phi_2.$$

Then $m(\Phi) > 0$.

Define the curve $L(\phi)$ in D_0 by

$$L(\phi) = \{z(\rho e^{i\phi}): 0 \le \rho < 1\} \qquad (\phi \in \Phi).$$

Then since $z(e^{i\phi})$ exists $(\phi \in \Phi)$, $L(\phi)$ ends at a point $\zeta(\phi)$; and since $|f(z(e^{i\phi}))| > \lambda$, $\zeta(\phi) \in F$. That is,

(4.6) $$L(\phi) \to \zeta(\phi) \in F \qquad (\phi \in \Phi).$$

Now let $E = \{\zeta(\phi): \phi \in \Phi\}$ and $E^* = \{e^{i\phi}: \phi \in \Phi\}$. Then

(4.7) $$m(E^*) > 0.$$

Now (4.4), (4.5), and (4.6) imply that

(4.8) $$E \subset A^*.$$

We shall prove that

(4.9) $$m_e(E) > 0.$$

Then, since the measurability of A^* follows from Theorem 1, the conclusion of the lemma will follow from (4.8).

The set

(4.10) $$U = \{\zeta \in C: \text{there exist arcs } \gamma_n \subset D_0 \text{ and } \gamma \subset C \text{ such that } \gamma_n \to \gamma \text{ and } \zeta \in \gamma^0\}$$

is open. Let S denote the countable set of endpoints of the components of U, and let $H = U \cup S$. We now prove that

(4.11) $$E \cap H \text{ is countable.}$$

It suffices to prove that $E \cap U$ is countable. Since U is covered by the arcs γ^0 in (4.10), it is covered by a countable subcollection of these arcs. Thus it suffices to show that if γ is one of the arcs in (4.10), then $E \cap \gamma^0$ contains at most one point. But this follows from the simple connectivity of D_0 and (4.3). Thus we have (4.11).

We note next that

(4.12) the correspondence $e^{i\phi} \to \zeta(\phi)$ $(\phi \in \Phi)$ is finite-to-one.

Suppose $L(\phi_1)$ and $L(\phi_2)$ ($\phi_1, \phi_2 \in \Phi$, $\phi_1 \neq \phi_2$) end at ζ. Let Δ be the interior of the Jordan curve $L(\phi_1) \cup L(\phi_2) \cup \{\zeta\}$. A theorem of Lindelöf states that if f is holomorphic and bounded in Δ, continuous on $L(\phi_1) \cup L(\phi_2) \cup \Delta$, and has limits as $z \to \zeta$ on $L(\phi_1)$ and $L(\phi_2)$, then f is continuous on $L(\phi_1) \cup L(\phi_2) \cup \Delta \cup \{\zeta\}$. It follows that $\Delta \cap \Gamma \neq \Box$. Since $\Gamma \cap (L(\phi_1) \cup L(\phi_2)) = \Box$ and D_0 is simply connected, some component of Γ must be contained in Δ and tend to ζ. Thus, since $|f(e^{i\phi})| > \lambda$ ($\phi \in \Phi$), we see that if infinitely many $L(\phi)$ ended at the same point ζ, f would have infinitely many distinct tracts ending at ζ, which implies that f would have infinitely many distinct tracts for ∞ ending at ζ (see [4, (2.6)]), contrary to assumption.

Thus from (4.11) and (4.12) we have the proposition

(4.13) $$\text{the set } \{e^{i\phi}: \zeta(\phi) \in H\} \text{ is countable.}$$

Let $E_1^* = E^* - \{e^{i\phi}: \zeta(\phi) \in H\}$. Then (4.7) and (4.13) imply that

(4.14) $$m(E_1^*) > 0.$$

Now $\{|z| = r\} - \Gamma$ $(0 < r < 1)$ is a finite union of open arcs of $\{|z| = r\}$ some of which, say $\Gamma_{r,i}$ $(i = 1, \cdots, n(r))$, lie in D_0. Let

$$w(\Gamma_{r,i}) = \gamma_{r,i} \qquad (0 < r < 1,\ i = 1, \cdots, n(r)).$$

Then $\gamma_{r,i}$ is a Jordan curve or a crosscut in $\{|w| < 1\}$. Let $D_{r,i}$ denote the component of $D_0 \cap \{r < |z| < 1\}$ that has $\Gamma_{r,i}$ on its boundary. Note that several of the Γ_r may be on the boundary of the same $D_{r,i}$. Let

$$w(D_{r,i}) = \Delta_{r,i} \qquad (0 < r < 1,\ i = 1, \cdots, n(r)).$$

In order to formulate definition (4.17), we let $\{\gamma_n\}$ be a sequence of the crosscuts γ_{r_n, i_n} such that

(4.15) $$r_n \uparrow 1$$

and

(4.16) $$\text{the origin and } \gamma_{n+1} \text{ are contained in different components of } \{|w| < 1\} - \gamma_n \ (n \geq 1).$$

Then $\{\gamma_n\}$ tends to a point or an arc of $\{|w| = 1\}$. Let

(4.17) $$H^* = \bigcup \{\gamma \text{ (closed arc): there exists } \{\gamma_n\} \text{ such that } \gamma_n \to \gamma\}.$$

We now prove that

(4.18) $$E_1^* \cap H^* = \Box.$$

Let $e^{i\phi} \in E^* \cap H^*$. Let γ be a (closed) arc containing $e^{i\phi}$ with the property that there exists a sequence $\{\gamma_n\}$ of the crosscuts γ_{r_n, i_n} satisfying (4.15), (4.16), and the relation $\gamma_n \to \gamma$. Let $\Gamma_n = z(\gamma_n)$. It follows from Koebe's lemma that the lengths of the Γ_n are bounded away from zero. Thus some subsequence of $\{\Gamma_n\}$ tends to an arc of C. Applying the theorem of Lindelöf, one easily shows that for each n neither end of γ_n is an endpoint of γ. Thus the radius at $e^{i\phi}$ meets each γ_n, and $L(\phi)$ meets each Γ_n. Therefore $\zeta(\phi) \in H$, and (4.18) is proved.

Let $\varepsilon > 0$, and let E_2^* be a closed subset of E_1^* such that

(4.19) $$m(E_2^*) > m(E_1^*) - \varepsilon.$$

Let $\Omega(w)$ be the harmonic measure in $\{|w| < 1\}$ of E_2^*, and let $\omega(z) = \Omega(w(z))$ $(z \in D_0)$. Then

(4.20) $$2\pi\omega(0) = 2\pi\Omega(0) = m(E_2^*).$$

Let G be an open subset of C containing E such that

(4.21) $$m(G) < m_e(E) + \varepsilon,$$

and let $\omega_1(z)$ be the harmonic measure in D of G. Let

$$u(z) = \omega_1(z) - \omega(z).$$

It is easy to see that

(4.22) $$\liminf_{\substack{z \to \zeta \\ z \in D_0}} u(z) \geq 0 \qquad (\zeta \in \Gamma \cup [F \cap G]).$$

Suppose now that $\zeta \in F \cap (C - G)$. In order to prove that

(4.23) $$\liminf_{\substack{z \to \zeta \\ z \in D_0}} u(z) \geq 0 \qquad (\zeta \in F \cap [C - G]),$$

it suffices to show that

(4.24) $$\lim_{\substack{z \to \zeta \\ z \in D_0}} \omega(z) = 0 \qquad (\zeta \in F \cap [C - G]).$$

Now let $\{z_n\}$ be any sequence of points in D_0 such that $z_n \to \zeta$, and choose $0 < r_n \uparrow 1$. To prove (4.24), it is sufficient to construct a subsequence of $\{z_n\}$ on which ω tends to zero. Now, for any $0 < r < 1$, there are only finitely many components of $\{r < |z| < 1\} \cap D_0$. Thus we may let D_1 be a component of $\{r_1 < |z| < 1\} \cap D_0$ that contains infinitely many z_n, and we may let D_{n+1} be a component of $\{r_{n+1} < |z| < 1\} \cap D_0$ contained in D_n and containing infinitely many z_n. The sequence $\{D_n\}$ thus defined has the properties

(4.25) $$D_{n+1} \subset D_n \qquad (n \geq 1)$$

and

(4.26) D_n contains infinitely many of the z_n $\qquad (n \geq 1)$.

Suppose also that $\Gamma \cap \{|z| = r_1\} \neq \square$. Let

$$w(D_n) = \Delta_n.$$

Then Δ_n is bounded by certain of the crosscuts $\gamma_{r_n, i}$ and a subset of $\{|w| = 1\}$. Let γ_n be the $\gamma_{r_n, i}$ that separates Δ_n from the origin. That is, let the origin and Δ_n be in different components of $\{|w| < 1\} - \gamma_n$. It follows from (4.25) that $\{\gamma_n\}$ satisfies (4.16). Let $\Gamma_n = z(\gamma_n)$. Using (4.25) and (4.26), we let L be a curve with initial point at the origin that is eventually in each D_n, passes through infinitely many of the z_n, and has nonvoid intersection with each Γ_n. Let

$$w(L) = L^*.$$

The curve L^* tends to a point ζ^* on an arc γ^* of $\{|w| = 1\}$. Suppose first that $L^* \to \gamma^*$. Then $\{\gamma_n\}$ tends to an arc γ^{**} containing γ^*. From (4.17) and (4.18) we see that the closed arc γ^* is contained in one of the components of $\{|w| = 1\} - E_2^*$. Thus Ω tends to zero on L^*.

Suppose now that

$$L^* \to \zeta^* = e^{i\phi}.$$

The curve L tends to a point ζ' or an arc γ of C. Suppose first that $L \to \gamma$. We wish to prove that $\zeta^* \notin E_1^*$. Suppose $\zeta^* \in E_1^*$. As before, we see that neither end of γ_n $(n \geq 1)$ can be ζ^*. Thus the radius at ζ^* must cross each γ_n, and $L(\phi)$ must cross each Γ_n. Since the connected set

$$L \cup \left(\bigcup_{n=1}^{\infty} \Gamma_n \right)$$

tends to an arc γ_1 containing γ and $\zeta(\phi) \in \gamma_1$, and since $\gamma \subset H$, we see that the relation $\zeta^* \in E_1^*$ has been contradicted. Thus $\zeta^* \notin E_1^*$ and Ω tends to zero on L^*.

Suppose finally that $L \to \zeta'$. Since $z_n \to \zeta$, we have the relation

$$\zeta' = \zeta \in F \cap (C - G). \tag{4.27}$$

Suppose $\zeta^* \in E^*$. If L^* crosses the radius at ζ^* in every neighborhood of ζ^*, then $\zeta(\phi) = \zeta$, which contradicts (4.27). Suppose then that in some neighborhood of ζ^*, L^* does not meet the radius at ζ^*. Let L^{**} be a simple curve in L^* that tends to ζ^*. Applying the theorem of Lindelöf again, we see that $z(L^{**})$ and $L(\phi)$ must end at the same point. Thus $\zeta' = \zeta(\phi)$, and (4.27) has been contradicted. Therefore $\zeta^* \notin E^*$, and Ω tends to zero on L^*.

Thus in any case Ω tends to zero on L^*, and ω tends to zero on L. Since L passes through infinitely many of the z_n, we see that (4.24) is proved. Now (4.22) and (4.23) imply

$$\omega(0) \leq \omega_1(0). \tag{4.28}$$

From (4.19), (4.20), (4.21), and (4.28) we have the inequality

$$m(E_1^*) < m_e(E) + 2\varepsilon. \tag{4.29}$$

Thus, since ε is arbitrary, (4.14) and (4.29) imply (4.9) and, as we have seen, the proof of Lemma 2 is complete.

Proof of Theorem 2. Suppose that there exists an f satisfying (4.1), (4.2), and

$$m(A^*) = 0. \tag{4.30}$$

It follows from (4.1), (4.30), and Lemma 2 that for each $\lambda > 0$, f is unbounded in each component of $\{|f| > \lambda\}$. Thus in each such component we can build a tract for ∞ (see MacLane [4, p. 26]). It follows from (4.1) that *for each* $\lambda > 0$, *there are only finitely many components of* $\{|f| > \lambda\}$.

Now let γ be an arc in C that does not meet the end of any arc tract of f for the value ∞. Choose $\zeta \in \gamma^0$. Using (4.30), we choose a sequence $\{z_n\}$ $(z_n \to \zeta)$ such that

$$|f(z_n)| \to \infty.$$

Let D_1 be a component of $\{|f| > 1\}$ containing infinitely many z_n. Let D_{n+1} be a component of $\{|f| > n+1\}$ that is contained in D_n and contains infinitely many z_n. The sequence $\{D_n\}$ thus defined determines a tract for ∞ whose end contains ζ. Thus we have a contradiction, and Theorem 2 is proved.

COROLLARY 1. *If* f *has only finitely many asymptotic tracts, then the ends of the arc tracts of* f *for the value* ∞ *cover* C. *In particular, if* f *has only one tract, then it is for the value* ∞, *and its end is* C.

Remarks. It is well known that every f has at least one asymptotic tract. Example 1 is an f with only one tract. Example 2 is an f with exactly two tracts, one of which is a point tract for the value zero.

COROLLARY 2. *If* f *has only finitely many tracts for* ∞, *then either* $m(A^*) > 0$ *or* f *has only finitely many tracts.*

Proof. Suppose f has only finitely many tracts for ∞, $m(A^*) = 0$, and f has infinitely many distinct tracts. Applying Theorem 2, we see that the ends of the arc tracts of f for ∞ cover C. Let $\zeta_1, \cdots, \zeta_n$ be the finite set of endpoints of the ends of the arc tracts of f for ∞. Then the end of each tract of f for a finite value must contain some ζ_i $(1 \leq i \leq n)$. Thus, since f has infinitely many tracts for finite values, infinitely many of the ends of these tracts must contain some fixed ζ_i, say ζ_1. But this implies that f has infinitely many distinct tracts for ∞ with end containing ζ_1, contrary to the assumption.

COROLLARY 3. *If* f *has no arc tracts and* $m(A^*) = 0$, *then* f *has infinitely many distinct tracts for* ∞.

Remarks. This behavior is illustrated by the modular function, which has only countably many tracts (all point tracts), and for which A_∞ is countable and dense. Indeed, the theorem of MacLane stated in Section 1 implies that if f satisfies the hypothesis of Corollary 3 and $f \in \mathscr{A}$, then A_∞ is dense. Example 4 shows that the conclusion of Corollary 3 cannot be "A_∞ is dense." Corollary 3 implies in particular that if all of the asymptotic tracts of f end at the same point (as in the case in Example 4), then f has infinitely many distinct tracts for ∞.

The following two theorems are simple consequences of what has already been proved.

THEOREM 3. *If* $w = f(z)$ *is unbounded and does not have the asymptotic value* ∞, *then for each* $N > 0$

$$m[A(\{|w| > N\})] > 0. \tag{4.31}$$

Remark. Example 3 shows that (4.31) does not follow from the assumptions that f is unbounded and satisfies (4.1) and (4.2).

Proof. Suppose that f satisfies the hypothesis of the theorem. If for any $\lambda > N$, f were unbounded in each component of $\{|f| > \lambda\}$, we could build a tract for ∞. Thus there exists a $\lambda > N$ such that f is bounded in some component of $\{|f| > \lambda\}$. It is clear that (4.31) follows from the proof of Lemma 2.

Now let V denote the set of point asymptotic values of f.

THEOREM 4. *If* f *satisfies* (4.1) *and* (4.2), *then* V *contains a closed set of positive harmonic measure.*

Remark. For a local version of Theorem 4 for $f \in \mathscr{A}$ see [4, p. 28].

Proof. A theorem of Priwalow [5, p. 210] states that if a function meromorphic in D has angular limits on a set F of positive measure, then the set of angular limit values of f at points of F contains a closed set of positive harmonic measure. Now, if for some $\lambda > 0$ f is bounded in some component of $\{|f| > \lambda\}$, then the function f(z(w)) in the proof of Lemma 2 has angular limits on E^*, and it follows from the theorem of Priwalow that V contains a closed set of positive harmonic measure. Suppose now that f satisfies (4.1) and (4.2), and that V contains no closed set of positive harmonic measure. Then for each λ, f is unbounded in each component of $\{|f| > \lambda\}$. It follows that for each λ, there are only finitely many components of $\{|f| > \lambda\}$. Now let γ and ζ be as in the proof of Theorem 2. Applying again the theorem of Priwalow, we choose a sequence $\{z_n\}$ $(z_n \to \zeta)$ such that $|f(z_n)| \to \infty$. Thus we get a contradiction, as in the proof of Theorem 2, and the proof of Theorem 4 is complete.

Now let

$$T^* = \{\zeta \in C : \zeta \text{ is the end of a point tract of f for a finite value}\}.$$

Then $T^* \subset A^*$. We shall prove that

$$A^* = T^* \cup \{\text{countable set}\}. \tag{4.32}$$

It will follow that *Theorem* 2 *remains valid with* A^* *replaced by* T^*. We say that *an arc tract of* f *yields a point asymptotic value at* $\zeta \in C$ *if there is a curve belonging to the tract* (see [4, p. 6]) *that tends to the point* ζ. It follows from Koebe's lemma that an arc tract for a finite value cannot yield point asymptotic values at two distinct points of C. Thus (4.32) follows from the following simple theorem.

THEOREM 5. *Each* f *can have only countably many arc tracts that yield point asymptotic values.*

Proof. Let

$$U = \{\zeta \in C : \gamma \text{ is the end of an arc tract of f and } \zeta \in \gamma^0\}. \tag{4.33}$$

Let S denote the countable set of endpoints of the components of the open set U. It is clear that not more than two distinct arc tracts of f can yield point asymptotic values at the same point of C. Thus only countably many arc tracts of f yield point asymptotic values at points of S, and it suffices to show that only countably many arc tracts yield point asymptotic values at points of U. Now, since U is covered by a countable number of the arcs γ^0 in (4.33), we need only observe that for a particular γ, all curves ending at interior points of γ on which f has a limit belong to the same tract of f. Thus Theorem 5 is proved.

5. EXAMPLES

Example 1. MacLane [4, Example 3] has constructed an $f \in \mathscr{A}$ such that

$$\min_{z \in J_n} |f(z)| \to \infty \quad (n \to \infty), \tag{5.1}$$

where $\{J_n\}$ is an expanding sequence of Jordan curves tending to C such that

$$M(r) \leq \frac{1}{1 - r} \quad (0 < r < 1), \tag{5.2}$$

where M(r) is the maximum modulus of f on $\{|z| = r\}$. It follows from (5.1) and a theorem of MacLane [4, Theorem 3] that f *has exactly one asymptotic tract and* $A_\infty = C$.

Example 2. Let f be the function of Example 1, and let

$$F(z) = (1 - z)^2 f(z).$$

Then from (5.2) we see that F *has the angular limit zero at* 1. Also, $f \in \mathscr{A}$, and F has an arc tract for ∞ with end C. It follows from (5.2) and a theorem of MacLane [4, Theorem 22] that F *has exactly two tracts.*

Example 3. Let F(z) be the function of Example 2. Let w(z) map

$$\Delta = D - \{z = x + iy: x \geq 0, \ y = 0\}$$

onto $\{|w| < 1\}$ one-to-one and conformally, and let z(w) be the inverse map. Let $\Phi(w) = F(z(w))$. Then Φ *has only one tract for* ∞; *its end is not* C. *Moreover, the set of finite asymptotic values of* Φ *is bounded.*

Example 4. We now construct an f *having asymptotic values only for point tracts that end at the point* 1. Choose

$$\frac{1}{2} < r_1 < \cdots < r_n \uparrow 1,$$

and let C_n be the circle with radius r_n contained in $D \cup C$ and tangent to C at 1. Let

$$T = D \cap \left[\{z: -1 < x \leq 0, \ y = 0\} \cup \left(\bigcup_{n=1}^{\infty} C_n\right)\right].$$

Let ϕ be a continuous function on T such that, for each n,

$$\phi(z) = 0 \ (z \in C_{2n}) \quad \text{and} \quad \phi(z) = 1 \ (z \in C_{2n-1}).$$

It follows from the tress argument of Bagemihl and Seidel [2, Theorem 1] applied to the "modified tress" T that there exists an f such that

$$|f(z) - \phi(z)| \to 0 \quad \text{as } |z| \to 1 \quad (z \in T).$$

It is clear that f has the desired property.

Example 5. Consider the function f of Example 4 in the half-disc

$$\triangle = D \cap \{z: x < 0\}.$$

Given $\varepsilon > 0$, let $w(z)$ be a one-to-one conformal map of $\triangle$ onto $\{|w| < 1\}$ so that $\{z: x = 0, -1 \leq y \leq 1\}$ corresponds to an arc γ of length ε, and let $z(w)$ be the inverse map. Let $F(w) = F(z(w))$. Then F *does not have the asymptotic value* ∞, *and the ends of all asymptotic tracts of* f *are contained in* γ.

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Rice University
and
University of Wisconsin—Milwaukee