REMARK ON A RESULT OF KAPLANSKY CONCERNING C(X)

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In this paper, let R be a chain (with more than one element) endowed with the interval topology, let X be a compact Hausdorff space, and let R(X) denote the family of continuous functions in R^X. For f, g $\in$ R(X), let $f \leq g$ mean that $f(x) \leq g(x)$ for all $x \in X$, and let $f < g$ mean that $f(x) < g(x)$ for all $x \in X$. For $f \in R(X)$, let G_f denote the graph of f in $X \times R$. Under the partial ordering $\leq$, R(X) is a lattice.

In [4] Kaplansky described all the lattice automorphisms of R(X) that are bicontinuous in the topology of uniform convergence (here R is the real line); if ϕ is such an automorphism, there exist a homeomorphism T of X onto X and a continuous mapping p of $X \times R$ into R such that $\phi(f)(Tx) = p(x, f(x))$ for all $x \in X$ and all $f \in R(X)$, and for each $x \in X$ the mapping $r \to p(x, r)$ is increasing. (Milgram [5] presents a similar result for the case where R(X) is regarded as a multiplicative semigroup.) He observed that if X satisfies the first countability axiom, then each automorphism ϕ of R(X) must be bicontinuous, and hence of this form. Finally, he presented a compact space X and an automorphism ϕ of R(X) that cannot be so described [4, p. 629].

We shall present analogues of these results in a much broader context in which Kaplansky's arguments do not apply (see Examples 1 and 2). The prime ideals employed in [3] and [4] will not enter our development of Theorems I, II, and III.

Definition 1. A sublattice L of R(X) is an R-sublattice if (1) for each $x \in X$, $Lx = \{f(x): f \in L\}$ consists of more than one element, and (2) given $f_1, f_2 \in L$, $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists an $h \in L$ such that $h(x_i) = f_i(x_i)$ for $i = 1, 2$.

Note that a characterizing sublattice of R(X) in the sense of Anderson and Blair [1] is an R-sublattice. For if

$$h_1(x_1) < f_1(x_1), \qquad h_1(x_2) > f_2(x_2), \qquad h_2(x_1) > f_1(x_1), \qquad h_2(x_2) < f_2(x_2),$$

then $(h_1 \vee f_1) \wedge (h_2 \vee f_2)$ coincides with f_1 at x_1 and with f_2 at x_2. On the other hand, an R-sublattice L is characterizing if and only if for each $x \in X$, Lx has no maximal or minimal element. (See Examples 1 and 2, and also compare R-sublattices with the c-characterizing lattices of Blair [2].)

Throughout this paper, L_1 and L_2 will be R-sublattices of R(X), and ϕ will be a lattice isomorphism of L_1 onto L_2.

Definition 2. The isomorphism ϕ of L_1 onto L_2 is increasing if for f, g $\in L_1$, $f < g$ if and only if $\phi(f) < \phi(g)$.

THEOREM I. *A necessary and sufficient condition that the isomorphism* ϕ *of* L_1 *onto* L_2 *be increasing is that there exist a homeomorphism* T *of* X *onto* X *and a mapping* p *of* $\bigcup_{f \in L_1} G_f$ *into* R, *continuous on each* G_f, *such that for each* $x \in X$, $r \to p(x, r)$ *is an increasing mapping of* $L_1 x$ *onto* $L_2(Tx)$, *and such that*

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$$\phi(f)(Tx) = p(x, f(x))$$

for all $f \in L_1$, $x \in X$.

THEOREM II. *If* ϕ *is an isomorphism of* L_1 *onto* L_2 *that is bicontinuous in the topology of pointwise convergence, then* ϕ *is increasing.*

THEOREM III (Kaplansky). *If* R *is a subset of the real line and* ϕ *is an isomorphism of* L_1 *onto* L_2 *that is bicontinuous in the uniform topology, then* ϕ *is increasing.*

THEOREM IV. *If* R *is the real line and* X *is either locally connected or sequentially compact, then each automorphism* ϕ *of* R(X) *is bicontinuous in the uniform topology and is increasing.*

Before constructing proofs we present some R-lattices to which the arguments given in [3] and [4] are inapplicable.

Example 1. Let R have a compact, totally disconnected order topology for which there exists a homeomorphism h of R into R mapping no element into itself. Let X be the same space, and let L consist of all $f \in R(X)$ such that for each $r \in R$, $f^{-1}(r)$ is at most a finite subset of X.

Obviously L is a lattice; we claim that L is an R sublattice of R(X). Note that h and the identity mapping on X show that Lx consists of at least two elements, for each $x \in X$. For $f_1, f_2 \in L$, $x_1, x_2 \in X$, and $x_1 \neq x_2$, let U_1, U_2 be complementary closed and open subsets of X such that $x_1 \in U_1$, $x_2 \in U_2$, and set $g = f_1$ on U_1 and $g = f_2$ on U_2; then $g \in L$ and $g(x_i) = f_i(x_i)$ for $i = 1, 2$. Thus L is an R-sublattice of R(X).

For any $r \in R$, $x \in X$, an argument like that in [4] would employ the boundary of the ideal $\{f \in L: f(x) \leq r\}$, say, in the topology of pointwise convergence. But in the present case this boundary may be void; indeed, it must be void if r is an isolated point in R. Since R contains a maximal and a minimal element, the argument in [3, Section 6] would require that for any disjoint closed sets $A, B \subset X$ some $f \in L$ is constant on A and B, respectively. But in the present case no function in L is constant on an infinite subset of X.

Example 2. Let E be a closed, totally disconnected, proper subset of a compact Hausdorff space X, and let R be the real line. Let L consist of all functions $f \in R(X)$ such that for each $x \in E$, f(x) is 0 or 1. By elementary topology and the Tietze Extension Theorem, it follows that L is an R-sublattice of R(X). The arguments in [3] and [4] again fail, as they do in Example 1.

We now develop proofs of Theorems I to IV. Since sufficiency in Theorem I is evident, we present no proof of it. To prove necessity in Theorem I, suppose until further notice that ϕ is increasing.

LEMMA 1. *There exists a unique one-to-one mapping* T *of* X *onto* X *such that for all* $f, g \in L_1$ *and all* $x \in X$, $f(x) = g(x)$ *if and only if* $\phi(f)(Tx) = \phi(g)(Tx)$.

Proof. First we observe that if $f_1, f_2 \in L_1$, $x_1, x_2 \in X$, and $x_1 \neq x_2$, then there exists an $h \in L_1$ such that $h(x_1) = f_1(x_1)$ and $h(x_2) \neq f_2(x_2)$. This is clear since there exist $g, h \in L_1$ such that

$$g(x_2) \neq f_2(x_2), \qquad h(x_1) = f_1(x_1), \qquad h(x_2) = g(x_2).$$

The corresponding assertion is also true of L_2.

Fix a function $f \in L_1$ and an element $x^* \in X$. Given any two functions $g, h \in L_1$ satisfying $g \leq f \leq h$ and $\phi(g)(x^*) = \phi(f)(x^*) = \phi(h)(x^*)$, let $C(h, g)$ denote the set of all points $u \in X$ for which $g(u) = f(u) = h(u)$. (For example, $C(f, f) = X$.) Plainly, $C(h, g)$ is compact and nonvoid; if $\phi(g) \not< \phi(h)$, then $g \not< h$. We claim that the collection of sets $C(h, g)$ over all such functions g, h has the finite intersection property. To show that $C(h_1, g_1) \cap C(h_2, g_2) \cap \cdots \cap C(h_n, g_n)$ is nonvoid, observe that if

$$h = \bigvee_{i=1}^{n} h_i \quad \text{and} \quad g = \bigwedge_{i=1}^{n} g_i,$$

then $g \leq f \leq h$, $\phi(g)(x^*) = \phi(f)(x^*) = \phi(h)(x^*)$, and $C(h, g)$ is nonvoid; but $g \leq g_i \leq f \leq h_i \leq h$ for $i = 1, \cdots, n$, and clearly $C(h, g) = \bigcap_{i=1}^{n} C(h_i, g_i)$.

It follows that there is a nonvoid compact subset C of X such that if $g, h \in L_1$, $g \leq f \leq h$, and $\phi(g)(x^*) = \phi(f)(x^*) = \phi(h)(x^*)$, then g, h, and f coincide on C. Now let g be any function in L_1 with $\phi(g)(x^*) = \phi(f)(x^*)$. It follows that $f \vee g$ and $f \wedge g$ coincide with f on C, and consequently g coincides with f on C.

Fix some $x \in C$. By a similar argument on ϕ^{-1}, there exists a nonvoid compact subset S of X such that if $g \in L_1$ and $g(x) = f(x)$, then $\phi(g)$ and $\phi(f)$ coincide on S. We claim that $S = \{x^*\}$. Indeed, if $x_1 \in S$ and $x_1 \neq x^*$, select $k \in L_2$ such that $k(x^*) = \phi(f)(x^*)$, $k(x_1) \neq \phi(f)(x_1)$; then $\phi^{-1}(k)(x) = f(x)$ and $k(x_1) = \phi(f)(x_1)$, which is impossible. Similarly we see that $C = \{x\}$.

We have thus far shown that for each $x^* \in X$ there exists a unique point $x \in X$ such that for $g \in L_1$, $f(x) = g(x)$ if and only if $\phi(f)(x^*) = \phi(g)(x^*)$. By the above argument on ϕ^{-1}, it follows that for each $x \in X$ there exists a unique point $x^* \in X$ such that for $g \in L_1$, $f(x) = g(x)$ if and only if $\phi(f)(x^*) = \phi(g)(x^*)$. Let T be the mapping of X into X defined by $Tx = x^*$. Then T is a one-to-one mapping of X onto X.

Select a function $f' \in L_1$, and let T' be the mapping determined by f' in the same way that T is determined by f; that is, for $g \in L_1$, let $f'(x) = g(x)$ if and only if $\phi(f')(T'x) = \phi(g)(T'x)$. To complete the proof of Lemma 1 it suffices to show that $T = T'$. The proof is by contradiction; suppose $x_1, x_2 \in X$, $x_1 \neq x_2$, and $Tx_1 = T'x_2$. Select $g_1, g_2 \in L_1$ such that

$$g_1(x_1) = f(x_1), \quad g_1(x_2) = f'(x_2), \quad g_2(x_1) = f(x_1), \quad g_2(x_2) \neq f'(x_2).$$

Then

$$\phi(f)(Tx_1) = \phi(g_1)(Tx_1) = \phi(g_1)(T'x_2) = \phi(f')(T'x_2)$$

and

$$\phi(f')(T'x_2) \neq \phi(g_2)(T'x_2) = \phi(g_2)(Tx_1) = \phi(f)(Tx_1),$$

which is impossible. Thus Lemma 1 is proved.

Construct the mapping p of $\bigcup_{f \in L_1} G_f$ into R as follows; for

$$(x, r) \in \bigcup_{f \in L_1} G_f,$$

let $p(x, r) = \phi(g)(Tx)$, where g is a function in L_1 satisfying $g(x) = r$. By Lemma 1, p is well defined.

LEMMA 2. *For any* $x \in X$, *the mapping* $r \to p(x, r)$ *is an increasing mapping of* $L_1 x$ *onto* $L_2(Tx)$.

Proof. Let $(x, r_1), (x, r_2) \in \bigcup_{f \in L_1} G_f$ and $r_1 < r_2$. Say $g_1, g_2 \in L_1$ and $g_1(x) = r_1$, $g_2(x) = r_2$. Then $(g_1 \vee g_2)(x) = r_2$ and $(g_1 \wedge g_2)(x) = r_1$, and because $\phi(g_1 \wedge g_2) \leq \phi(g_1 \vee g_2)$, it follows that

$$p(x, r_1) = \phi(g_1 \wedge g_2)(Tx) \leq \phi(g_1 \vee g_2)(Tx) = p(x, r_2).$$

By Lemma 1, $\phi(g_1 \wedge g_2)(Tx) \neq \phi(g_1 \vee g_2)(Tx)$ and $p(x, r_1) < p(x, r_2)$.

LEMMA 3. T *is bicontinuous.*

Proof. Select any $x \in X$, and let U be a neighborhood of x. Choose (x, r_1) and (x, r_2) in $\bigcup_{f \in L_1} G_f$, with $r_1 < r_2$. For any $x_1 \in X - U$ there exist functions $g, h \in L_1$ such that $g(x) = r_1$, $h(x) = r_2$, and $h(x_1) < g(x_1)$. By a simple compactness argument it follows that there are functions $g_1, \cdots, g_n \in L_1$, $h_1, \cdots, h_n \in L_1$ such that

$$(h_1 \wedge \cdots \wedge h_n)(x) = r_2, \qquad (g_1 \vee \cdots \vee g_n)(x) = r_1,$$

and $g_1 \vee \cdots \vee g_n$ exceeds $h_1 \wedge \cdots \wedge h_n$ on $X - U$. For convenience of notation, set

$$g_0 = g_1 \vee \cdots \vee g_n, \qquad h_0 = h_1 \wedge \cdots \wedge h_n.$$

By Lemmas 1 and 2 we see that

$$Tx \in X(\phi(g_0) < \phi(h_0)) \quad \text{and} \quad T^{-1}X(\phi(g_0) < \phi(h_0)) \subset U.$$

Therefore T^{-1} is continuous, and since X is a compact Hausdorff space, T is also continuous.

Proof of Theorem I. In Lemmas 1 to 3 we produced a homeomorphism T of X onto X and a mapping p from $\bigcup_{f \in L_1} G_f$ into R such that $\phi(f)(Tx) = p(x, f(x))$ for all $f \in L_1, x \in X$. We showed that for each $x \in X$ the mapping $r \to p(x, r)$ is increasing. It remains only to show that p is continuous on each G_f. If $f \in L_1$, the mappings $(x, f(x)) \to Tx$ and $x \to \phi(f)(x)$ are continuous functions of G_f into X and X into R, respectively, and the composite mapping $(x, f(x)) \to \phi(f)(Tx) = p(x, f(x))$ is also continuous. This concludes the proof.

Observe also that if ϕ is increasing, the mappings T and p in Theorem I are unique. This is implicit in our argument.

Proof of Theorem II. Assume all the hypotheses of Theorem II. It suffices to show that if $f, g \in L_1$ and $\phi(f) < \phi(g)$, then $f < g$. The proof is by contradiction (the converse statement can be proved similarly). Assume $f \not< g$. Since ϕ is an isomorphism, clearly $f \leq g$ and $f(x) = g(x)$ for some $x \in X$. Without loss of generality we may assume that $L_1 x$ contains a point $r > f(x)$. (A dual argument disposes of the case where $r < f(x)$.)

For any points $x_1, \cdots, x_n \in X$ distinct from x, there exist functions $f_1, \cdots, f_n \in L_1$ such that $f_i(x) = r$ and $f_i(x_i) = f(x_i)$ for all i. Then

$$\bigwedge_{i=1}^{n} (f_i \vee f)(x) = r \qquad \text{and} \qquad \bigwedge_{i=1}^{n} (f_i \vee f)(x_i) = f(x_i)$$

for all i. For every neighborhood U of f in L_1 we can produce a function $g_U \in L_1$ such that $g_U(x) = r$ and $g_U \wedge g \in U$. Since $g(x) < g_U(x)$, we see that $\phi(g_U) \not< \phi(g)$ and $X(\phi(g) \leq \phi(g_U))$ is a compact nonvoid subset of X. Likewise, for any neighborhoods $U_1, \cdots, U_m$ of f in L_1,

$$g(x) < \left(\bigwedge_{j=1}^{m} g_{U_j}\right)(x), \qquad X\left[\phi(g) \leq \bigwedge_{j=1}^{m} \phi(g_{U_j})\right] = \bigcap_{j=1}^{m} X[\phi(g) \leq \phi(g_{U_j})],$$

and this intersection is nonvoid. It follows that there exists a point $y \in X$ such that

$$\phi(f)(y) < \phi(g)(y) \leq \phi(g_U)(y) \qquad \text{and} \qquad \phi(g)(y) = \phi(g_U \wedge g)(y)$$

for each neighborhood U of f in L_1. This contradicts the bicontinuity of ϕ, and Theorem II is proved.

It is also worth noting that for each $x \in X$, the mapping $r \to p(x, r)$ is bicontinuous. To see this, select a neighborhood U of f in L_1. There is a neighborhood J of f(x) in R such that to each $r \in J \cap (L_1 x)$ there corresponds a $g_U \in U$ such that $g_U(x) = r$; the construction of g_U is essentially like the construction in the preceding paragraph. From the bicontinuity of ϕ it follows that the mapping $r \to p(x, r)$ is continuous in r. Reversal of the roles of L_1 and L_2 shows that this mapping is bicontinuous.

Proof of Theorem III. As in the proof of Theorem II, assume that

$$f(x) = g(x), \qquad \phi(f) < \phi(g), \qquad r \in L_1 x, \qquad r > f(x).$$

Select $\varepsilon > 0$, and put $W = X(g - f \geq \varepsilon)$; then $x \in X - W$. By compactness and by a preceding argument, there exist functions $h_1, \cdots, h_n \in L_1$ such that

$$W \subset \bigcup_{i=1}^{n} X(h_i < f + \varepsilon)$$

and $h_i(x) = r$, for $i = 1, \cdots, n$. Set

$$k = (h_1 \vee f) \wedge (h_2 \vee f) \wedge \cdots \wedge (h_n \vee f).$$

Then $k(x) = r$ and $f \leq k < f + \varepsilon$ on W. Hence $|k \wedge g - f| \leq \varepsilon$.

Then, given any neighborhood U of f in L_1, we can produce a function $g_U \in L_1$ such that $g_U(x) = r$ and $g_U \wedge g \in U$. The argument can be completed as in the proof of Theorem II.

We make one further comment regarding Theorem III. In general, the order topology on R might not coincide with the metric topology on R, and the functions in L_1 might not be continuous mappings of X into the real line. We claim that if all the functions in L_1 are continuous in this latter sense, then p is a continuous mapping of $\bigcup_{f \in L_1} G_f$ into the real line (compare with [4, Lemma 2]). To see this, suppose that $f, g \in L_1$, $\delta > 0$, and $x \in X$ satisfy the condition $|f(x) - g(x)| < \delta$.

By an argument essentially like that in the preceding paragraph, there exists an $h \in L_1$ such that $h(x) = g(x)$ and $|h - f| \leq \delta$. It follows from the bicontinuity of ϕ that to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that for any $x \in X$, $|p(x, r) - p(x, f(x))| \leq \varepsilon$ if $|r - f(x)| < \delta$. From this and the continuity of f we see that p is continuous at $(x, f(x))$ for all $x \in X$, $f \in L_1$.

Proof of Theorem IV. Let ϕ be a lattice automorphism of R(X). By [4] there is a homeomorphism T of X onto X such that for $f, g \in R(X)$, $x \in X$, the relation $f(x) < g(x)$ implies $\phi(f)(Tx) \leq \phi(g)(Tx)$ and $\phi(f)(Tx) < \phi(g)(Tx)$ implies $f(x) \leq g(x)$.

First we show by contradiction that for $f, g \in R(X)$, $\phi(f) < \phi(g)$ implies $f < g$. Assume that $\phi(f) < \phi(g)$ and $f \not< g$. Clearly $f \leq g$, and $X(f < g)$ and $X(f = g)$ are nonvoid sets. We claim that $X(f = g)$ has void interior; for otherwise there would be a function $h \in R(X)$, coinciding with f on $X(f < g)$, such that $f \leq h$ and h exceeds f at some point in the interior of $X(f = g)$. Then $h \wedge g = f$, $\phi(h) \wedge \phi(g) = \phi(f)$, and $\phi(h)$ exceeds $\phi(f)$ at some point, which is impossible.

Assume that X is locally connected, and select $x \in X(f = g)$. Then x is in the closure of $X(f < g)$, and if U is any connected neighborhood of x, $(g - f)(U)$ is a connected subset of R containing an interval with left endpoint 0 and positive right endpoint. For any neighborhood U of x, the set $(g - f)(U)$ contains such an interval, because X is locally connected. For each positive integer n, set $E_n = X(g - f = 1/n)$; then each neighborhood U of x intersects E_n for all but finitely many n. Define the function h_0 on the set $\left(\bigcup_{n=1}^{\infty} E_n\right) \cup X(f = g)$ by the rule

$$h_0 = \begin{cases} f = g & \text{on } X(f = g), \\ g + 1/n & \text{on } E_n \text{ (n odd)}, \\ f - 1/n & \text{on } E_n \text{ (n even)}. \end{cases}$$

Routine arguments show that h_0 is continuous on $\left(\bigcup_{n=1}^{\infty} E_n\right) \cup X(f = g)$. By the Tietze Extension Theorem, some $h \in R(X)$ coincides with h_0 on this set.

Every neighborhood of x contains points y and z with $h(y) < f(y)$, $g(z) < h(z)$. Thus in every neighborhood of Tx there exist Ty and Tz with $\phi(h)(Ty) \leq \phi(f)(Ty)$, $\phi(g)(Tz) \leq \phi(h)(Tz)$. But $\inf[\phi(g) - \phi(f)] > 0$, because X is compact. This conflicts with the continuity of $\phi(h)$ at Tx. It follows that ϕ is increasing in the locally connected case.

Now assume that X is sequentially compact. Since $X(f = g)$ is not open, there exists a sequence $\{x_n\}$ of points in X such that $f(x_n) < g(x_n)$ for all n and $\lim_{n\to\infty}[g(x_n) - f(x_n)] = 0$. Let $\{x_{n_i}\}$ be a subsequence converging to some point $x \in X(f = g)$. Conclude the argument as above, employing $\{x_{n_i}\}$ in lieu of the sequence $\{E_n\}$. It follows that ϕ is increasing in the sequentially compact case.

Finally, to show that ϕ is uniformly bicontinuous in either case, select $\varepsilon > 0$ and $f \in R(X)$. Then $\phi(f - \varepsilon) < \phi(f) < \phi(f + \varepsilon)$, and if g is a function in R(X) such that $\phi(f - \varepsilon) < \phi(g) < \phi(f + \varepsilon)$, we have the inequalities $f - \varepsilon < g < f + \varepsilon$. The conclusion is evident.

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