ON THE EXISTENCE OF ALMOST PERIODIC MOTIONS

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In the first part of this paper we investigate the question of necessary and sufficient conditions for the existence of (Bohr) almost periodic motions in dynamical systems. The results are applied, in the second part, to give an existence criterion for almost periodic solutions of ordinary differential equations.

PART I. DYNAMICAL SYSTEMS

V. V. Nemyckiĭ and V. V. Stepanov [7] investigated conditions under which the ω-limit set of a positively Lagrange-stable motion is a minimal set of almost periodic motions. They proved that a sufficient condition for this to hold is that 1) the motion approximates the ω-limit set uniformly and 2) the motion is uniformly positively Lyapunov-stable with respect to the positive semitrajectory. They remarked that the question of necessary conditions is still open.

In this part of the paper we present another sufficient condition (Theorem 5) that the ω-limit set of a positively Lagrange-stable motion be a minimal set of almost periodic motions. It will be seen that our condition (which is stronger than that of Nemyckiĭ and Stepanov) is more readily applicable. The question of a necessary condition remains open. However, we are able to give a partial solution to this problem.

Let (X, d) be a metric space with a dynamical system π: $X \times R \to X$, where R denotes the real numbers. Let Ω_p denote the ω-limit set of the point $p \in X$. Let $\gamma^+(p) = \{\pi(p, t): t \geq 0\}$ denote the positive semitrajectory and $\gamma(p) = \{\pi(p, t): t \in R\}$ the trajectory of the motion through p. If $I \subset R$, then $\pi(p, I)$ will denote the set $\{\pi(p, t): t \in I\}$. If $A \subset X$ and $\varepsilon > 0$, then the ball about A of radius ε is given by

$$\mathfrak{B}(A; \varepsilon) = \{p \in X: d(p, A) < \varepsilon\}.$$

A motion $\pi(p, t)$ is said to be *recurrent* if for every $\varepsilon > 0$ there is a $T > 0$ such that for each $t_0 \in R$

$$\gamma(p) \subset \mathfrak{B}(\pi(p, [t_0, t_0 + T]); \varepsilon).$$

The motion $\pi(p, t)$ is *Lagrange-stable* if $\mathrm{Cl}\,\gamma(p)$ is compact, and it is *positively Lagrange-stable* if $\mathrm{Cl}\,\gamma^+(p)$ is compact. It is known [7, Theorem 7.09] that a Lagrange-stable motion $\pi(p, t)$ is recurrent if and only if for every $\varepsilon > 0$ the set $\{\tau \in R: d(p, \pi(p, \tau)) < \varepsilon\}$ is relatively dense in R. A motion $\pi(p, t)$ is said to be (Bohr) *almost periodic* if for every $\varepsilon > 0$ the set

$$\{\tau \in R: d(\pi(p, t), \pi(p, t + \tau)) < \varepsilon \text{ for all } t \text{ in } R\}$$

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is relatively dense in R. It is known [7, Theorem 8.02] that every almost periodic motion is recurrent.

G. D. Birkhoff [2] showed that, in a complete metric space, a nonempty, compact set E is the closure of a recurrent motion if and only if E is a minimal set; that is, if and only if E is a nonempty, closed, invariant set containing no proper subset with these three properties. (The completeness of the space is needed only in the proof that recurrence of $\pi(p, t)$ implies that $\mathrm{Cl}\,\gamma(p)$ is minimal.) Now, if a dynamical system contains a positively Lagrange-stable motion $\pi(p, t)$, then Ω_p is a nonempty, compact, invariant set. Consequently, Ω_p contains a minimal set and there is a recurrent motion in the dynamical system. This can be summarized in the following existence theorem.

THEOREM 1. *Given a dynamical system on a complete metric space* (X, d), *the following statements are equivalent:*

(A) *There is a recurrent motion in the dynamical system.*

(B) *There is a positively Lagrange-stable motion in the dynamical system.*

Thus, the existence of a positively Lagrange-stable motion always implies the existence of a recurrent motion. One may therefore ask what further assumptions are needed to insure the existence of an almost periodic motion. An answer to this is given in the following theorem of Nemyckiĭ and Stepanov [7, Theorem 9.06].

THEOREM 2. *Assume that* $\pi(p, t)$ *is positively Lagrange-stable. If* (i) $\pi(p, t)$ *approximates* Ω_p *uniformly and* (ii) $\pi(p, t)$ *is uniformly positively Lyapunov-stable with respect to* $\gamma^+(p)$, *then* Ω_p *is a minimal set of almost periodic motions.*

(We recall that $\pi(p, t)$ *approximates* Ω_p *uniformly* provided that to each $\varepsilon > 0$ there corresponds a $T > 0$ such that $\Omega_p \subset \mathfrak{B}(L; \varepsilon)$ for each arc L of $\gamma^+(p)$ whose (time) length is greater than T. Further, recall that a motion $\pi(p, t)$ is *uniformly positively Lyapunov-stable with respect to a set* $D \subset X$ provided that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d(\pi(q, t), \pi(\hat{q}, t)) < \varepsilon$ for all $t \geq 0$, whenever $q \in \gamma^+(p)$, $\hat{q} \in D$, and $d(q, \hat{q}) < \delta$.

Before stating the main results, we recall some known theorems. The proofs of these statements can be found in Nemyckiĭ and Stepanov [7].

THEOREM 3. *Assume that* $\pi(p, t)$ *is positively Lagrange-stable. A necessary and sufficient condition that* Ω_p *be a minimal set is that* $\pi(p, t)$ *approximates* Ω_p *uniformly.*

THEOREM 4. *Let* E *be the closure of an almost periodic motion. If the metric space* X *is complete, then for every* $p \in E$ *the motion* $\pi(p, t)$ *is uniformly (positively) Lyapunov-stable with respect to* E.

We can now prove the following extension of Theorem 2.

THEOREM 5. *Assume that* $\pi(p, t)$ *is a positively Lagrange-stable motion. A sufficient condition that* Ω_p *be a minimal set of almost periodic motions is that* $\pi(p, t)$ *be uniformly positively Lyapunov-stable with respect to* $\gamma^+(p) \cup \Omega_p$.

Proof. In order to show that the condition is sufficient we shall make use of Theorems 2 and 3 and prove that if $\pi(p, t)$ is uniformly positively Lyapunov-stable with respect to $\gamma^+(p) \cup \Omega_p$, then Ω_p is a minimal set; that is, $\pi(p, t)$ approximates Ω_p uniformly. Thus, we want to show that if $q, r \in \Omega_p$, then $r \in \mathrm{Cl}\,\gamma(q)$. In other words, we want to show that for every $\varepsilon > 0$ there is a τ in R such that $d(r, \pi(q, \tau)) < 2\varepsilon$.

Since $q, r \in \Omega_p$, there exist sequences $\{s_n\}$ and $\{t_n\}$ in R with $s_n \to \infty$, $t_n \to \infty$ and such that

$$q_n = \pi(p, s_n) \to q \quad \text{and} \quad r_n = \pi(p, t_n) \to r.$$

With $\varepsilon > 0$ given, choose $\delta > 0$ by the condition of uniform positive Lyapunov-stability. Then $d(q_m, q) < \delta$ for sufficiently large m, say $m \geq M$, so that by the Lyapunov-stability $d(\pi(q, t), \pi(q_m, t)) < \varepsilon$ for all $t \geq 0$. Now let $m \geq M$ be fixed. Then

$$r_n = \pi(p, t_n) = \pi(q_m, t_n - s_m)$$

for all n. Since $t_n \to \infty$, there is an N_1 such that $t_n - s_m > 0$ for $n \geq N_1$. This in turn implies that

$$d(\pi(q, t_n - s_m), r_n) = d(\pi(q, t_n - s_m), \pi(q_m, t_n - s_m)) < \varepsilon$$

for $n \geq N_1$. Now choose N_2 so that $d(r_n, r) < \varepsilon$ whenever $n \geq N_2$. Pick some $N \geq \max(N_1, N_2)$ and set $\tau = t_N - s_m$. Then

$$d(\pi(q, \tau), r) \leq d(\pi(q, t_N - s_m), r_N) + d(r_N, r) < 2\varepsilon,$$

and, hence, Ω_p is a minimal set of almost periodic motions. Q. E. D.

We see then that our condition (Theorem 5) is stronger than the condition of Nemyckiǐ and Stepanov (Theorem 2). One may ask whether either of the conditions is necessary. The answer to this is not known; however, we are able to give a partial answer.

For this, it is necessary to distinguish between two cases: 1) $\Omega_p \cap \gamma^+(p) \neq \emptyset$, that is, $\pi(p, t)$ is positively Poisson-stable, and 2) $\Omega_p \cap \gamma^+(p) = \emptyset$, that is, $\pi(p, t)$ is positively asymptotic.

If Ω_p is a minimal set of almost periodic motions, then, in the first case, $\gamma^+(p)$ lies in the invariant set Ω_p and the motion $\pi(p, t)$ is almost periodic. Consequently, by Theorem 4, $\pi(p, t)$ is uniformly positively Lyapunov-stable with respect to

$$\mathrm{Cl}\,\gamma(p) = \Omega_p = \gamma^+(p) \cup \Omega_p.$$

(The hypotheses of Theorem 4 require X to be complete. In our application, let us restrict ourselves to the compact, invariant set $\mathrm{Cl}\,\gamma(p)$. Since this set is compact, it is complete, and Theorem 4 applies. However, the conclusion of Theorem 4 is a local condition, so that it remains valid in the original dynamical system.) We have thus proved the following result.

COROLLARY. *Let* $\pi(p, t)$ *be positively Lagrange-stable; then the following statements are equivalent.*

(A) $\pi(p, t)$ *is an almost periodic motion.*

(B) $\pi(p, t)$ *is positively Poisson-stable and uniformly positively Lyapunov-stable with respect to* $\mathrm{Cl}\,\gamma^+(p)$.

For the second case, where $\pi(p, t)$ is positively asymptotic, the following is the best that we can prove now.

LEMMA. *Let* $\pi(p, t)$ *be a positively Lagrange-stable motion. If* $\pi(p, t)$ *is positively asymptotic, then* $\pi(p, t)$ *is positively Lyapunov-stable with respect to* $\gamma^+(p) \cup \Omega_p$.

(A motion $\pi(p, t)$ is said to be *positively Lyapunov-stable with respect to a set* $D \subset X$ if for every $q \in \gamma^+(p)$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d(\pi(q, t), \pi(\hat{q}, t)) < \varepsilon$$

for all $t \geq 0$, whenever $\hat{q} \in D$ and $d(q, \hat{q}) < \delta$.)

Proof. Assume that $\pi(p, t)$ is not positively Lyapunov-stable with respect to $\gamma^+(p) \cup \Omega_p$. (There is no loss of generality in assuming this holds at p.) Then there exist an $\varepsilon > 0$ and sequences $\{p_n\} \subset \gamma^+(p) \cup \Omega_p$ and $\{t_n\} \subset R$ such that

$$t_n \geq 0, \qquad p_n \to p, \qquad d(\pi(p, t_n), \pi(p_n, t_n)) \geq 3\varepsilon.$$

Since $\pi(p, t)$ is positively asymptotic and $p_n \to p$, we can choose the $\{p_n\}$ to lie in $\gamma^+(p)$; that is, there is a sequence $\{s_n\}$ such that $p_n = \pi(p, s_n)$. Since p does not lie in Ω_p, the sequence $\{s_n\}$ is bounded. If the sequence $\{s_n\}$ has an accumulation point $s > 0$, then a subsequence converges to s, say $s_n \to s$, and

$$p_n = \pi(p, s_n) \to \pi(p, s) = p.$$

That is, $\pi(p, t)$ is periodic, and this contradicts the assumption that $\pi(p, t)$ is positively asymptotic. Consequently, $\lim s_n = 0$.

Using the continuity with respect to initial conditions, one can easily verify that the sequence $\{t_n\}$ has no finite accumulation point; in other words, $t_n \to \infty$ as $n \to \infty$.

We now extract a subsequence of $\{\pi(p, t_n)\}$ and denote it by $\{\pi(p, t_n)\}$, so that $\pi(p, t_n) \to q \in \Omega_p$. By continuity, it then follows that $\pi(p, t_n + t) \to \pi(q, t)$ for every $t \in R$. Moreover, this convergence is uniform if we restrict t to lie in compact intervals in R.

By continuity we can choose $\tau > 0$ so that $d(q, \pi(q, t)) < \varepsilon$ whenever $|t| \leq \tau$. With τ fixed, choose N_1 so that $d(\pi(p, t_n + t), \pi(q, t)) < \varepsilon$ when $n \geq N_1$ and $|t| \leq \tau$, and choose N_2 so that $|s_n| \leq \tau$ if $n \geq N_2$. If $n \geq \max(N_1, N_2)$, we have the contradiction

$$\begin{aligned} 3\varepsilon &\leq d(\pi(p, t_n), \pi(p, t_n + s_n)) \\ &\leq d(\pi(p, t_n), q) + d(q, \pi(q, s_n)) + d(\pi(q, s_n), \pi(p, t_n + s_n)) < 3\varepsilon. \end{aligned}$$

Hence $\pi(p, t)$ is positively Lyapunov-stable with respect to $\gamma^+(p) \cup \Omega_p$. Q.E.D.

Remarks. 1. It follows that for every point $q \in \gamma^+(p)$ the motion $\pi(q, t)$ is positively Lyapunov-stable with respect to $\gamma^+(p) \cup \Omega_p$. The uniformity of the Lyapunov-stability would follow [7, Theorem 8.04] if we could show that for each point q in Ω_p the motion $\pi(q, t)$ is positively Lyapunov-stable with respect to $\gamma^+(p) \cup \Omega_p$.

2. If our condition (Theorem 5) is both a necessary and sufficient condition for Ω_p to be a minimal set of almost periodic motions, then it is equivalent to the condition of Nemyckiĭ and Stepanov (Theorem 2).

3. Even though our condition that Ω_p be a minimal set of almost periodic motions is stronger than that of Nemyckiĭ and Stepanov, it is simpler, and, consequently, it appears to be more readily applicable. We shall illustrate this with an application of Theorem 5 in the theory of ordinary differential equations.

PART II. ORDINARY DIFFERENTIAL EQUATIONS

We now consider the ordinary differential equation

$$x' = f(x, t) \tag{1}$$

where $f: R^n \times R^1 \to R^n$ is continuous and periodic in t, say of period 1. We shall assume that (1) satisfies some uniqueness condition. If $p = (x_p, t_p) \in R^n \times R^1$, let $\phi(p, t)$ denote the unique solution of (1) passing through p. If $x_0, x_1 \in R^n$, let $|x_0 - x_1|$ denote the Euclidean distance between x_0 and x_1.

In 1950, J. L. Massera [5] considered the question: under what conditions does the existence of a bounded solution imply the existence of a periodic solution of period 1? He showed that the implication holds, with no further assumptions, if the dimension n is 1. (The proof in this case is a simple application of the Poincaré-Bendixson theory.) For $n = 2$, the implication holds under the further assumption that all of the solutions of (1) are defined for all $t \geq 0$. For the case $n = 2$, an example is constructed in [5] to show that a stronger hypotheses is needed. However, the differential equation in this example does have a solution of period 2.

For higher dimensions ($n \geq 2$) it seems that the following question is more appropriate: when does the existence of a bounded solution imply the existence of an almost periodic solution? Using Theorem 5 we can now prove the following result.

THEOREM 6. *If there exists a solution* $\phi(p, t)$ *of* (1) *that is bounded for* $t \geq t_p$ *and is uniformly stable, then there exists an almost periodic solution of* (1).

(Recall that a solution $\phi(p, t)$ is said to be *uniformly stable* (in the sense of Persidskiĭ) if (i) $\phi(p, t)$ can be continued for all $t \geq t_p$ and (ii) to every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that for each $\tau \geq t_p$ the inequality $|\phi(p, t) - \phi(q, t)| < \varepsilon$ holds for all $t \geq \tau$, whenever $|\phi(p, \tau) - \phi(q, \tau)| < \delta$. Also, a solution $\phi(p, t)$ is said to be *almost periodic* if $\phi(p, t)$ can be continued for all t in R and, furthermore, if for every $\varepsilon > 0$ the set

$$\{\tau \in R: |\phi(p, t + \tau) - \phi(p, t)| < \varepsilon \text{ for all } t \text{ in } R\}$$

is relatively dense in R.)

Before proving this theorem, we note the related work of L. L. Helms and C. R. Putnam [3], [8], [9], who considered the autonomous differential equation

$$x' = f(x) \tag{2}$$

on R^n with

$$\operatorname{div} f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = 0. \tag{3}$$

In [9], Putnam showed that if $\phi(p, t)$ is a (future) bounded, stable solution of (2), then $\phi(p, t)$ itself is almost periodic. Insofar as Putnam's result is viewed as an existence criterion (it is actually more than this), we see that the above theorem is more general.

We now prove Theorem 6. Since the vector field $f(x, t)$ is periodic in t, we can identify the hyperplanes $t = 0$ and $t = 1$ and in this way map the space $R^n \times R^1$ onto the open torus (or annulus) $T = R^n \times S^1$, where S^1 is the one-dimensional sphere. This mapping we shall denote by $(x, t) \to (x, [t])$, where $[t] = t \pmod 1$. If $p = (x_p, t_p) \in R^n \times R^1$, let $\mathfrak{p} = (x_p, [t_p])$ denote the image in T. Since the vector field $f(x, t)$ is periodic of period 1, we know that if $p = (x_0, t_0)$ and $q = (x_0, t_0 + k)$, where k is any integer, then $\phi(p, t) = \phi(q, t + k)$ for all t. In other words, if p and q are both pre-images of some point $\mathfrak{p}$ in T (that is, $\mathfrak{p} = \mathfrak{q}$), then

$$\phi(p, t_p + t) = \phi(q, t_q + t)$$

for all t.

A metric can be defined on the sphere S^1 in several ways. For the sake of definiteness we shall let $\rho([s], [t])$ denote the metric inherited from the plane after S^1 is mapped onto the circle of circumference 1.

If $\mathfrak{p} = (x_p, [t_p])$ and $\mathfrak{q} = (x_q, [t_q])$ denote points in T, then

$$d(\mathfrak{p}, \mathfrak{q}) = |x_p - x_q| + \rho([t_p], [t_q])$$

is a metric on T. Furthermore, the metric topology on T is the same as the topology T inherits as a subset of R^{n+1}.

Let us now formally define the mapping $\pi(\mathfrak{p}, t)$ as

$$\pi(\mathfrak{p}, t) = (\phi(p, t_p + t), [t_p + t]), \tag{4}$$

where $\mathfrak{p} \in T$ and p is any pre-image of $\mathfrak{p}$. (We have already observed that the right side of (4) depends only on $\mathfrak{p}$ and t.) The mapping π is defined only formally, because we have not yet specified the domain or range. The domain is a subset of $T \times R^1$, and the range is in T. We shall now show that there exists a subset $X \subset T$ such that $\pi: X \times R^1 \to X$ and π is a flow or dynamical system on X. (It will be evident that the set X we define below is maximal with respect to the above property.)

Define the three sets

$$LB^+ = \{p \in R^n \times R^1 : \phi(p, t) \text{ is defined for all } t \geq t_p\},$$

$$LB^- = \{p \in R^n \times R^1 : \phi(p, t) \text{ is defined for all } t \leq t_p\},$$

$$LB = LB^+ \cap LB^- .$$

Let X denote the image of the set LB under the given mapping of $R^n \times R^1$ onto T. One can easily verify that if the set X (or LB) is not empty, then (4) defines a flow on X.

LEMMA. *If there is a solution* $\phi(p, t)$ *of* (1) *that is bounded for all* $t \geq t_p$, *then the set* LB *is not empty.*

Proof. Let $\phi(p, t)$ be a solution of (1) that is bounded for $t \geq t_p$. We shall now define the limit set of $\mathfrak{p}$ and show that it is nonempty and lies in X.

Let $\gamma^+(\mathfrak{p}) = \{\pi(\mathfrak{p}, t): t \geq 0\}$. Since $\phi(p, t)$ is bounded, the set $\gamma^+(\mathfrak{p})$ is relatively compact, that is, $\mathrm{Cl}\,\gamma^+(\mathfrak{p})$ is compact. Also, the set $\gamma^+(\pi(\mathfrak{p}, \tau))$ is relatively compact for $\tau \geq 0$. Furthermore, the family of compact sets $\{\mathrm{Cl}\,\gamma^+(\pi(\mathfrak{p}, \tau))\}$ $(\tau \geq 0)$ is decreasing; hence the intersection

$$L_{\mathfrak{p}} = \bigcap_{\tau \geq 0} \mathrm{Cl}\,\gamma^+(\pi(\mathfrak{p}, \tau)) \tag{5}$$

is nonempty and compact. We call this the limit set of $\mathfrak{p}$.

Now observe that the limit set $L_{\mathfrak{p}}$ can be characterized as the collection of all points $\mathfrak{q}$ in T such that $\mathfrak{q} = \lim \pi(\mathfrak{p}, t_n)$ for some sequence $\{t_n\}$ with $t_n \to \infty$. Since $L_{\mathfrak{p}}$ is not empty, let $\mathfrak{q} \in L_{\mathfrak{p}}$. We want to show that if t is any element of R, then $\pi(\mathfrak{q}, t)$ is defined and lies in $L_{\mathfrak{p}}$.

Choose $t_n \to \infty$ so that $\mathfrak{p}_n = \pi(\mathfrak{p}, t_n) \to \mathfrak{q}$. Let p_n be the pre-image of $\mathfrak{p}_n$ with $0 \leq t_{p_n} < 1$. Then the sequence $\{t_{p_n}\}$ converges to a limit t_0, where $0 \leq t_0 \leq 1$. Now let q be the pre-image of $\mathfrak{q}$ with $t_q = t_0$; then $p_n \to q$ in $R^n \times R^1$. Thus, if I is any interval on which $\phi(q, t)$ is defined, then for each t in I there is an N such that $\phi(p_n, t)$ is defined for $n \geq N$ and, moreover, $\phi(p_n, t) \to \phi(q, t)$ as $n \to \infty$. (This is well known. It can be proved in several ways. For example, it is a direct application of Corollary 3.1 of [10].) Consequently,

$$\pi(\mathfrak{p}_n, t) = \pi(\mathfrak{p}, t_n + t) \to \pi(\mathfrak{q}, t).$$

Furthermore, the point $\pi(\mathfrak{q}, t)$ lies in the compact set $L_{\mathfrak{p}}$, since $(t_n + t) \to \infty$ as $n \to \infty$. Now let I be the maximal interval of definition of the solution $\phi(q, t)$. Kamke's theorem [4] asserts that either $I = R^1$ or $|\phi(q, t)| \to \infty$ as $t \to$ bdry I. Since $\pi(\mathfrak{q}, t)$ lies in a compact set for each t in I, the set $\{|\phi(q, t)|: t \in I\}$ is bounded, and hence $I = R^1$.

Since $L_{\mathfrak{p}} \subset X$, both X and LB are nonempty, and this completes the proof of the lemma.

As we noted above, the function π defines a flow on X. Moreover, if $p \in LB$ (or $\mathfrak{p} \in X$), the set $L_{\mathfrak{p}}$ is the ω-limit set of $\mathfrak{p}$, that is, $L_{\mathfrak{p}} = \Omega_{\mathfrak{p}}$.

If $q \in R^n \times R^1$ is any pre-image of $\mathfrak{q}$, then

$$|\phi(q, t + \tau) - \phi(q, t)| \leq d(\pi(\mathfrak{q}, t + \tau), \pi(\mathfrak{q}, t)).$$

Consequently, if $\pi(\mathfrak{q}, t)$ is an almost periodic motion in X, then the corresponding solution $\phi(q, t)$ is an almost periodic solution of (1). Using a theorem of Besicovitch, one can show that the converse is also true.

(We note that $\pi(\mathfrak{q}, t)$ is a periodic motion in X if and only if the corresponding solution $\phi(q, t)$ is a periodic solution of (1) with integral period. If, for example, $\phi(q, t)$ were a periodic solution of (1) with an irrational period, then the corresponding motion $\pi(\mathfrak{q}, t)$ in X would be almost periodic, but not periodic.)

Continuing with the proof of the theorem, we let $\phi(p, t)$ be the solution of (1) satisfying the hypotheses of the theorem. If $p \in LB$, then $\mathfrak{p} \in X$, and the motion $\pi(\mathfrak{p}, t)$ is positively Lagrange-stable and uniformly positively Lyapunov-stable with respect to X. Consequently, this motion is uniformly positively Lyapunov-stable with respect to $\gamma^+(\mathfrak{p}) \cup \Omega_{\mathfrak{p}}$. By Theorem 5, every motion in $\Omega_{\mathfrak{p}}$ is almost periodic; therefore, if $\mathfrak{q} \in \Omega_{\mathfrak{p}}$, then the solution $\phi(q, t)$ (where q denotes some pre-image of $\mathfrak{q}$) is almost periodic.

If p does not lie in LB, that is, if $\phi(p, t)$ is not defined for all $t \leq t_p$, then there are several ways of modifying the argument. We shall outline one method. Since $\mathrm{Cl}\,\gamma^+(\mathfrak{p})$ is compact, there exists a $\rho > 0$ such that $\mathrm{Cl}\,\gamma^+(\mathfrak{p}) \subset \mathfrak{B}(\rho) \times S^1$, where $\mathfrak{B}(\rho)$ is the set of x in R^n with $|x| < \rho$. Now consider the equation

$$x' = \alpha(|x|)f(x, t), \tag{6}$$

where $\alpha(r)$ is a real-valued C^∞-function, defined on $0 \leq r < \infty$ and satisfying the condition

$$\alpha(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \rho, \\ 0 & \text{if } 2\rho \leq r < \infty. \end{cases}$$

The solutions of (6) are unique, and they are defined for all t. Moreover, they agree with those of (1) in the region $\mathfrak{B}(\rho) \times R^1$. If we now examine the associated dynamical systems, we see that the sets $L_\mathfrak{p}$ and the motions in each set are the same. Also, for equation (6), $\mathfrak{p} \in X$ and $L_\mathfrak{p} = \Omega_\mathfrak{p}$. Consequently, every motion in $L_\mathfrak{p}$ is almost periodic, and if $\mathfrak{q} \in L_\mathfrak{p}$, then the corresponding solution $\phi(q, t)$ of (1) is almost periodic. This completes the proof of the theorem.

Example. One may ask whether the assumption of stability can be dropped. The answer is negative even in the case of autonomous differential equations. This is seen in the following example, which was suggested by Lawrence Markus.

In [1] it is shown that there exists a 3-dimensional, compact manifold M and a C^1-vector field f(x) on M such that M itself is a minimal set, in the flow defined by f(x), and such that there is no almost periodic motion in M.

The manifold M can be imbedded in R^7, and the vector field f(x) admits a C^1-extension $\hat{f}(x)$ to all of R^7. Furthermore, since M is compact, this extension can be chosen so that the solutions of the equation $x' = \hat{f}(x)$ are defined for all t.

Now consider R^8, and let $y \in R^8$ be written as $y = (x, z)$, where $x \in R^7$ and $z \in R^1$. The manifold M now lies in the hyperplane $\{(x, 0): x \in R^7\}$. Consider

$$y' = F(y) \tag{7}$$

on R^8, where $F(y) = (\hat{f}(x), d(y, M))$ and $d(y, M)$ is the distance between y and M. The function F is (Lipschitz) continuous, and the solutions of (7) are unique. Furthermore, the set M is the only nonempty ω-limit set. Consequently (7) has no almost periodic motions.

Remarks. 1. R. K. Miller pointed out that, under the hypotheses of Theorem 6, every almost periodic solution of (1) in the pre-image of $L_\mathfrak{p}$ is uniformly stable.

2. Theorem 6 can be extended to the case in which f(x, t) is defined on $W \times R^1$, where W is an open set in R^n. Essentially the same argument can be used. In this case one would have to assume that the solution $\phi(t)$ lies in a compact subset in W.

3. R. K. Miller [6] has made an application of Theorem 5 to ordinary differential equations with almost periodic right sides.

4. In a forthcoming paper [11], the second author has continued this analysis to establish a relationship between periodic solutions and asymptotic stability.

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