NOTE ON AN INVARIANT OF KERVAIRE

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In [2] Kervaire defined the so-called Arf invariant $\Phi(M) \in Z_2$ for an $(n-1)$-connected, compact, closed, C^∞ manifold M of dimension 2n, where n is odd and $n \neq 1, 3, 7$ (see also [3]). In fact, he showed that Φ induces a homomorphism from the $2n^{th}$ framed cobordism group into Z_2, and that $\Phi = 0$ if $n = 5$. It is an unsolved problem whether $\Phi = 0$ for all n.

Let $\Omega_m(\mathrm{Spin})$ denote the m^{th} spin or cobordism group [4]. The aim of this note is to generalize Φ in the following sense. We define a homomorphism

$$\Psi: \Omega_{2n}(\mathrm{Spin}) \to Z_2$$

for $n \equiv 1 \pmod 4$ such that $\Psi(M) = \Phi(M)$ if M is as above. The writer has not been able to show that $\Psi \neq 0$. It is known that the image of framed cobordism in Spin cobordism is not zero. (Milnor has shown that there exists a homotopy 10-sphere that is not a Spin boundary.)

In the following, $n \equiv 1 \pmod 4$, $n > 1$, and all cohomology groups have Z_2 coefficients. Recall that

$$Sq^{n+1} = Sq^2 Sq^{n-1} + Sq^1 Sq^2 Sq^{n-2}.$$

Hence, on n-dimensional cohomology classes,

$$Sq^2 Sq^{n-1} + Sq^1 (Sq^2 Sq^{n-2})$$

is a relation. In [1] it is shown that such a relation gives rise to a secondary cohomology operation

$$\psi: H^n(X) \cap \mathrm{Ker}\, Sq^{n-1} \cap \mathrm{Ker}\, Sq^2 Sq^{n-2} \to H^{2n}(X)/Sq^2 H^{2n-2}(X) + Sq^1 H^{2n-1}(X).$$

Furthermore, if $\psi(u)$ and $\psi(v)$ are defined, then $\psi(u+v)$ is defined and

$$\psi(u+v) = \psi(u) + \psi(v) + u \cup v$$

modulo the indeterminacy of the operation.

Suppose M is a closed, compact, simply connected 2n-manifold such that the Stiefel-Whitney class $W_2(M)$ is zero. If $u \in H^n(M)$, then

$$Sq^{n-1} u \in H^{2n-1}(M) = 0, \qquad Sq^2 Sq^{n-2} u = W_2 Sq^{n-2} u = 0,$$

$$Sq^2 H^{2n-2}(M) = W_2 H^{2n-2}(M) = 0, \qquad \text{and} \qquad Sq^1 H^{2n-1}(M) = 0.$$

Hence ψ defines a quadradic function

$$\psi: H^n(M) \to H^{2n}(M).$$

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Let

$$\Psi(M) = \sum \psi(\lambda_i)(M)\psi(\mu_i)(M),$$

where $\lambda_i, \mu_i \in H^n(M)$ $(i = 1, 2, \cdots r)$ is a basis such that $\lambda_i \lambda_j = \mu_i \mu_j = 0$ and $\lambda_i \mu_j = \delta_{ij}$ (compare with c(M) in [3, p. 535]). $\Psi(M)$ is the Arf invariant of ψ and is independent of the choice of basis.

THEOREM. *ψ induces a homomorphism $\Psi: \Omega_{2n}(\mathrm{Spin}) \to Z_2$ such that $\Psi(M) = \Phi(M)$ if M is an (n - 1)-connected π-manifold.*

Proof. It is trivial to verify that Ψ is additive with respect to connected sums. Thus to show that Ψ is defined on $\Omega_{2n}(\mathrm{Spin})$ it is sufficient to verify that $\Psi(M) = 0$ if $M = \partial N$, where $W_1(N) = W_2(N) = 0$. Applying surgery (see [3]) to N, we make it 2-connected. Let $j: M \to N$ be the inclusion map. Recall that if $u \in H^n(N)$ and $v \in H^n(M)$, then $\delta(j^*(u)v) = u(\delta v)$, where $\delta: H^n(M) \to H^{n+1}(N, M)$. Using this fact, the cohomology exact sequence for (N, M), and Poincaré duality, one may choose elements $\bar{\lambda}_i \in H^n(N)$ and $\mu_j \in H^n(M)$ such that $j^*\bar{\lambda}_i$ and μ_j is a basis as above. Since N is 2-connected and M is 1-connected, $H^{2n-i}(N) \approx H_{i+1}(N, M) = 0$ for $i = 0$ and $i = 1$. Therefore ψ is defined and zero on $\bar{\lambda}_i$, and hence

$$\psi(j^*\bar{\lambda}_i) = j^*\psi(\bar{\lambda}_i) = 0.$$

Therefore $\Psi(M) = 0$.

Finally, to show that $\Psi(M) = \Phi(M)$ if M is an (n - 1)-connected π-manifold, we verify that ψ satisfies Lemma 8.3 of [3]. That is, we verify that an embedded n-sphere in M has trivial normal bundle if and only if its dual cohomology class v satisfies $\psi(v) = 0$. Let ν be this normal bundle, let $T(\nu)$ be the Thom space of ν, let $u \in H^n(T(\nu))$ be the Thom class, and let $f: M \to T(\nu)$ be the canonical map. Since $f^*(u) = v$, $\psi(v) = f^*\psi(u)$. But $T(\nu)$ is $S^2 \vee S^{2n}$ or $S^n \cup_{[\iota, \iota]} e^{2n}$ (where $\iota \in \pi_n(S^n)$ is the generator), according as ν is trivial or not. In [1] it is shown that $\psi(u) = 0$ in the first case and $\psi(u) \neq 0$ in the second case.

REFERENCES

1. E. H. Brown and F. P. Peterson, *Whitehead products and cohomology operations*, Quart. J. Math. Oxford Ser. (2) 15 (1964), 116-120.

2. M. A. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. 34 (1960), 257-270.

3. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres: I*, Ann. of Math. (2) 77 (1963), 504-537.

4. J. Milnor, *Spin structures on manifolds*, Enseignement Math. 9 (1963), 198-203.

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