THE INTEGRAL REPRESENTATION RING OF A FINITE GROUP

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1. INTRODUCTION

We shall be concerned with matrix representations of a finite group G by nonsingular matrices with entries in a ring R, where R is a discrete valuation ring of characteristic zero. Let $\Gamma = RG$, the group ring of G with coefficients in R. We may then, equivalently, consider left Γ-modules having finite R-bases.

Let us assume that the Krull-Schmidt theorem is valid for such Γ-modules, that is, that every Γ-module is uniquely expressible as a direct sum of indecomposable modules. (Toward the end of this section we shall give some sufficient conditions for the validity of the Krull-Schmidt theorem.) Then we may define the *integral representation ring* $A(\Gamma)$, as follows. Denote by $\{M\}$ the isomorphism class of the Γ-module M. Form the additive abelian group generated by the symbols $\{M\}$ ranging over the distinct isomorphism classes of Γ-modules, with defining relations

$$\{M\} = \{M'\} + \{M''\} \qquad \text{whenever } M \cong M' \oplus M''.$$

On this additive group we impose a ring structure, by defining

$$\{M\}\{M'\} = \{M \otimes_R M'\},$$

where the action of G on $M \otimes_R M'$ is given (as is customary) by

$$g(m \otimes m') = gm \otimes gm' \qquad (g \in G,\ m \in M,\ m' \in M').$$

The ring thus obtained we denote by $A(\Gamma)$, and call it the integral representation ring of Γ. Clearly, $A(\Gamma)$ is a commutative associative ring. Its unity element is $\{R\}$; here, R denotes the *trivial* Γ-module, that is,

$$g\alpha = \alpha \qquad (g \in G,\ \alpha \in R).$$

The above representation ring is an analogue of the modular representation algebra recently introduced by J. A. Green [4]. Let k be a field of characteristic p (where $p \geq 0$), and let Ω denote the complex field. Form the Ω-algebra $A_\Omega(kG)$ consisting of the Ω-linear combinations of symbols corresponding to the isomorphism classes of kG-modules, with relations and multiplication defined in the manner above. If p does not divide the order of G, then the group algebra kG is semisimple, and Green showed easily that the representation algebra $A_\Omega(kG)$ is also semisimple. Indeed, if $p = 0$, then $A_\Omega(kG)$ is isomorphic to the (commutative) algebra of generalized characters (with coefficients from Ω). But this latter algebra has no nonzero nilpotent elements; for if η is a generalized character such that $\eta^m = 0$

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for some m, then obviously $\eta = 0$. If $p \neq 0$ and p does not divide the order of G, the same reasoning applies if we use Brauer characters in place of ordinary characters.

Green also established a much more difficult result: *For any cyclic group* G, *the modular representation algebra* $A_\Omega(kG)$ *is semisimple.* Since $A_\Omega(kG)$ is a commutative algebra, this is equivalent to the assertion that $A_\Omega(kG)$ does not contain any nonzero nilpotent elements.

Here we investigate the analogous question: Does $A(\Gamma)$ contain nonzero nilpotent elements? Our main result is as follows:

THEOREM. *Let* G *be a cyclic group of order* n. *Let* R *be a discrete valuation ring of characteristic zero, with maximal ideal* P. *Suppose that the Krull-Schmidt theorem holds for* RG*-modules. Assume that* $n \in P^2$, *and if* $2 \in P$, *assume further that* $n \in 2P$. *Then the integral representation ring* A(RG) *contains at least one nonzero nilpotent element.*

On the other hand, it is possible to choose G *and* R *such that* A(RG) *contains no nonzero nilpotent element.*

Throughout this paper, we let Z_p denote the p-adic valuation ring in the rational field Q, and Z_p^* the ring of p-adic integers in the p-adic completion of Q.

COROLLARY. *The representation ring* $A(Z_p G)$ *contains nonzero nilpotent elements if* G *is a cyclic group of order* p^e *with* $e > 1$.

A trivial observation should be made at this point. Let G be an arbitrary finite group whose order is a unit in the discrete valuation ring R. Let K be the quotient field of R. There is then a one-to-one correspondence between the isomorphism classes of RG-modules (having finite R-bases) and the isomorphism classes of KG-modules (see [3, Theorem 76.17]). This correspondence induces an isomorphism $A(RG) \cong A(KG)$. But A(KG) is a subring of the commutative semisimple algebra $A_\Omega(KG)$, hence contains no nonzero nilpotent elements. The same is therefore true for A(RG).

To conclude this section, we list several conditions, any one of which implies the Krull-Schmidt theorem for RG-modules.

i) The order of G is a unit in R.

ii) R is a complete discrete valuation ring.

iii) The quotient field of R is an algebraic number field which is a splitting field for G.

iv) G is an arbitrary p-group, where $R = Z_p$.

For the proofs that each of these imply the Krull-Schmidt theorem, we cite the following references: for i), see [3, Theorem 76.17]; for ii), see [2], [10], or [8]; for iii), see [5]. In order to show that iv) is a sufficient condition for the validity of the Krull-Schmidt theorem, one first uses the Witt-Berman theorem [3, Theorem 42.8] to show that an irreducible QG-module remains irreducible upon extension of the ground field from Q to its p-adic completion. The desired result now follows as in [3, Lemma 76.28 and Theorem 76.29].

2. TENSOR PRODUCTS OF MODULES

Let G be an arbitrary finite group of order n, and let R be a discrete valuation ring of characteristic zero. Let $P = \pi R$ be the maximal ideal of R, and set $\overline{R} = R/P$ ($\overline{R}$ is then a field of characteristic p). Set $\Gamma = RG$, $\overline{\Gamma} = \overline{R}G$, and consider finitely generated left Γ-modules. Since we shall need to work with Γ-modules that do not necessarily have R-bases, Γ-modules having R-bases will be called *R-free* *Γ-modules*.

Assume hereafter that the Krull-Schmidt theorem is valid for R-free Γ-modules. Let X be a fixed R-free Γ-module satisfying

(1) $$P\Gamma \subset X \subset \Gamma .$$

If Y is an arbitrary R-free Γ-module, we shall show how the problem of calculating the tensor product module $Y \otimes_R X$ can be reduced to a calculation involving only $\overline{\Gamma}$-modules. Indeed, we shall see that (for fixed X) the isomorphism class of $Y \otimes_R X$ depends upon $\overline{Y} = Y/PY$ rather than upon Y.

Set $A = \Gamma/X$, so that there is an exact sequence of Γ-modules:

(2) $$0 \to X \to \Gamma \to A \to 0 .$$

From (1) we conclude that $PA = 0$, and thus that A may be viewed as $\overline{\Gamma}$-module.

Now let Y be any R-free Γ-module, and let $m = (Y{:}\ R)$ be the number of elements in an R-basis for Y. Set $\overline{Y} = Y/PY$, so that

$$0 \to PY \to Y \to \overline{Y} \to 0$$

is exact. Then

$$PY \otimes_R A \to Y \otimes_R A \to \overline{Y} \otimes_R A \to 0$$

is also exact. However, the image of $PY \otimes_R A$ in $Y \otimes_R A$ is zero, since $PA = 0$. This shows that $Y \otimes_R A \cong \overline{Y} \otimes_R A$. But both $\overline{Y}$ and A are $\overline{R}$-modules, which readily implies that $\overline{Y} \otimes_R A \cong \overline{Y} \otimes_{\overline{R}} A$. Thus

$$Y \otimes_R A \cong \overline{Y} \otimes_{\overline{R}} A \quad \text{as } \Gamma\text{-modules.}$$

Since Y is R-free, we obtain from (2) the exact sequence of Γ-modules

(3) $$0 \to Y \otimes_R X \to Y \otimes_R \Gamma \to Y \otimes_R A \to 0 .$$

Now $Y \otimes_R \Gamma \cong \Gamma^{(m)}$, where $\Gamma^{(m)}$ denotes the direct sum of m copies of Γ (see Swan [10, Lemma 5.1]). Thus (3) may be written as

(4) $$0 \to Y \otimes_R X \to \Gamma^{(m)} \to \overline{Y} \otimes_{\overline{R}} A \to 0 .$$

Let

$$\overline{Y} \otimes_{\overline{R}} A \cong B_1 \oplus \cdots \oplus B_t ,$$

where the B_i are indecomposable $\overline{\Gamma}$-modules. Each B_i is expressible as a quotient of a free $\overline{\Gamma}$-module, hence also as a quotient of a free Γ-module. Thus for each i $(1 \leq i \leq t)$, there is an exact sequence

$$0 \to M_i \to \Gamma^{(n_i)} \to B_i \to 0,$$

for some n_i and some R-free Γ-module M_i. Therefore the sequence

$$(5) \qquad 0 \to M_1 \oplus \cdots \oplus M_t \to \Gamma^{(n_1+\cdots+n_t)} \to B_1 \oplus \cdots \oplus B_t \to 0$$

is exact.

SCHANUEL'S LEMMA (see Swan [11]). *If*

$$0 \to M \to L \to B \to 0 \quad \textit{and} \quad 0 \to M' \to L' \to B' \to 0$$

are two exact sequences of Γ*-modules, where both* L *and* L' *are projective, and* $B \cong B'$, *then*

$$M \oplus L' \cong M' \oplus L.$$

Applying this lemma to the sequences (4) and (5), we obtain

$$(6) \qquad Y \otimes_R X \oplus \Gamma^{(n_1+\cdots+n_t)} \cong M_1 \oplus \cdots \oplus M_t \oplus \Gamma^{(m)}.$$

Since the Krull-Schmidt theorem is assumed to hold for Γ-modules, we may use (6) to calculate $Y \otimes_R X$. Note that (for fixed X) the result depends only on m and $B_1, \cdots, B_t$, that is to say, only on the $\overline{\Gamma}$-module $\overline{Y}$.

Later (in Section 5) we shall systematically use this approach to calculate certain tensor product modules. For the moment, we shall content ourselves with the following simple consequence of formula (6).

If X and Y are a pair of R-free Γ-modules such that

$$(7) \qquad P\Gamma \subset X \subset \Gamma, \quad P\Gamma \subset Y \subset \Gamma, \quad \overline{X} \cong \overline{Y},$$

then the preceding discussion implies that

$$X \otimes_R X \cong X \otimes_R Y \cong Y \otimes_R X \cong Y \otimes_R Y.$$

Hence in $A(\Gamma)$ we have the relation

$$(\{X\} - \{Y\})^2 = \{X \otimes X\} - \{X \otimes Y\} - \{Y \otimes X\} + \{Y \otimes Y\} = 0.$$

Thus, if X and Y are a pair of nonisomorphic Γ-modules satisfying (7), then $\{X\} - \{Y\}$ is a nonzero nilpotent element of the integral representation ring $A(\Gamma)$.

3. CYCLIC GROUPS

We keep the notation of Section 2, and we assume throughout this section that G is a cyclic group generated by an element g of order n. Embed R in the group ring Γ by the usual map $\alpha \in R \rightarrow \alpha \cdot 1 \in \Gamma$, where 1 is the identity element of G.

Let us choose X to be the ideal generated by π and $g - 1$ in the commutative ring Γ. Then $P\Gamma \subset X \subset \Gamma$, and X has an R-basis $\{x_1, \cdots, x_n\}$, where

$$x_1 = g - 1,\ x_2 = g^2 - 1,\ \cdots,\ x_{n-1} = g^{n-1} - 1,\ x_n = \pi.$$

We find at once that

$$gx_i = x_{i+1} - x_1 \ (1 \leq i \leq n - 2), \quad gx_{n-1} = -x_1, \quad gx_n = x_n + \pi x_1. \tag{8}$$

Consequently $\overline{X} = X/PX = \overline{U} \oplus \overline{R}\overline{x}_n$, where $\overline{U} = \Sigma_{i=1}^{n-1} \overline{R}\overline{x}_i$, with the action of g on $\overline{U}$ given by the first $n - 1$ equations in (8), with the x_i replaced by $\bar{x}_i$. (Indeed, $\overline{U}$ is isomorphic to the augmentation ideal of $\overline{\Gamma}$.) In the matrix representation of G afforded by the module $\overline{U}$, the matrix corresponding to g is

$$C = \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Let us put $h(\lambda) = \Sigma_{k=0}^{n-1} \lambda^k$, where λ is an indeterminate over $\overline{R}$. Then C is similar to the companion matrix of $h(\lambda)$, which shows that

$$\overline{U} \cong \overline{R}[\lambda]/(h(\lambda)),$$

where g acts on the right-hand module as multiplication by λ.

We shall now show that if n satisfies the hypotheses of the theorem given in Section 1, then the trivial $\overline{\Gamma}$-module $\overline{R}$ cannot be a $\overline{\Gamma}$-direct summand of $\overline{U}$. Thus, we assume that $n \in P^2$, and if $2 \in P$, we assume in addition that $n \in 2P$. This guarantees that $n(n - 1)/2 \in P$ in all cases.

Suppose that $\overline{R}$ is a $\overline{\Gamma}$-direct summand of $\overline{U}$. Then there is a direct sum decomposition:

$$\frac{\overline{R}[\lambda]}{(h(\lambda))} \cong \frac{\overline{R}[\lambda]}{(\lambda - 1)} \oplus \frac{\overline{R}[\lambda]}{(k(\lambda))}. \tag{9}$$

It follows at once that $(\lambda - 1)k(\lambda) = h(\lambda)$, so that

$$k(\lambda) = \lambda^{n-2} + 2\lambda^{n-3} + \cdots + (n - 2)\lambda + (n - 1) \in \overline{R}[\lambda].$$

But then $k(1) = n(n - 1)/2 = 0$ in $\overline{R}$, and thus $\lambda - 1$ is a factor of $k(\lambda)$. From (9) we then conclude that $k(\lambda)$ annihilates $\overline{R}[\lambda]/(h(\lambda))$, which is obviously impossible. Thus, the assumptions about n imply that $\overline{R}$ is not a $\overline{\Gamma}$-direct summand of $\overline{U}$.

Next, define $s = 1 + g + \cdots + g^{n-1}$, and choose Y as the ideal generated by π and s in the ring Γ. Then $P\Gamma \subset Y \subset \Gamma$, and Y has an R-basis $\{y_1, \cdots, y_n\}$, where

$$y_1 = \pi, \quad y_2 = \pi g, \quad \cdots, \quad y_{n-1} = \pi g^{n-2}, \quad y_n = s.$$

Note that

$$y_1 + y_2 + \cdots + y_{n-1} + \pi g^{n-1} = \pi y_n.$$

We see at once that

$$gy_i = y_{i+1} \ (1 \leq i \leq n-2), \quad gy_{n-1} = \pi y_n - (y_1 + \cdots + y_{n-1}), \quad gy_n = y_n.$$

Therefore $\overline{Y} = Y/PY = \overline{W} \oplus \overline{R}\bar{y}_n$, where $g\bar{y}_n = \bar{y}_n$, and where $\overline{W} = \Sigma_{i=1}^{n-1} \overline{R}\bar{y}_i$. The action of G on $\overline{W}$ is given by

$$g\bar{y}_i = \bar{y}_{i+1} \quad (1 \leq i \leq n-2), \qquad g\bar{y}_{n-1} = -(\bar{y}_1 + \cdots + \bar{y}_{n-1}).$$

The matrix corresponding to g (acting on $\overline{W}$) is

$$D = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix},$$

which is the transpose of the companion matrix of $h(\lambda)$. Hence C and D are similar over $\overline{R}$, which shows that $\overline{W} \cong \overline{U}$ as $\overline{\Gamma}$-modules.

We have thus shown that $\overline{X} \cong \overline{Y} \cong \overline{U} \oplus \overline{R}$, so that (7) holds. Our next task is to verify that X is not Γ-isomorphic to Y. We set $U = \Sigma_{i=1}^{n-1} Rx_i$ (this is a Γ-submodule of X; indeed, U is the augmentation ideal of Γ). Then U is an R-direct summand of X, and X/U is isomorphic to the trivial Γ-module R.

We shall determine a Γ-submodule V of Y such that

$$K \otimes_R V \cong K \otimes_R U, \qquad Y/V \cong R,$$

with V an R-direct summand of Y. The composition factors of the KG-module KG are the trivial module K, occurring with multiplicity 1, together with the composition factors of $K \otimes_R U$. Hence $\mathrm{Hom}_{KG}(K \otimes_R U, K) = 0$. Therefore any Γ-homomorphism of X into Y must induce a Γ-homomorphism of U into V. In particular, if $X \cong Y$ as Γ-modules, then $U \cong V$ as Γ-modules, and therefore $\overline{U} \cong \overline{V}$ as $\overline{\Gamma}$-modules. We shall show that this is impossible when n satisfies the hypotheses of the theorem in Section 1, since we shall verify that the trivial $\overline{\Gamma}$-module $\overline{R}$ is a $\overline{\Gamma}$-direct summand of $\overline{V}$. We may thus conclude that X is not isomorphic to Y, and hence that $\{X\} - \{Y\}$ is a nonzero nilpotent element of $A(\Gamma)$.

Since $n \in P^2$, we may write $n = \pi \cdot \pi'$ for some $\pi' \in P$. Now let V be the R-free P-module with basis $\{v_1, \cdots, v_{n-1}\}$, where

$$v_i = y_{i+1} - y_1 = \pi(g^i - 1) \qquad (1 \le i \le n-2),$$

$$v_{n-1} = \pi' y_1 - y_n = n - s = -\{(g-1) + (g^2-1) + \cdots + (g^{n-1}-1)\}.$$

Then V is an R-pure Γ-submodule of Y, hence an R-direct summand of Y. We have the relation

$$K \otimes_R V = \sum_{i=1}^{n-1}{}^{\oplus} K(g^i - 1),$$

so that $K \otimes_R V$ is the augmentation ideal of KG, and thus

$$K \otimes_R V \cong K \otimes_R U.$$

But then

$$K \otimes (Y/V) \cong (K \otimes Y)/(K \otimes V) \cong KG/(K \otimes_R U) \cong K,$$

which shows that $Y/V \cong R$. We see at once that

$$gv_i = v_{i+1} - v_1 \qquad (1 \le i \le n-3),$$

$$gv_{n-2} = -(2v_1 + v_2 + v_3 + \cdots + v_{n-2} + \pi v_{n-1}),$$

$$gv_{n-1} = \pi' v_1 + v_{n-1}.$$

Therefore $\overline{V} = V/PV = (\Sigma_{i=1}^{n-2} \overline{R}\bar{v}_i) \oplus \overline{R}\bar{v}_{n-1}$, $g\bar{v}_{n-1} = \bar{v}_{n-1}$, and so the trivial $\overline{\Gamma}$-module $\overline{R}$ is a $\overline{\Gamma}$-direct summand of $\overline{V}$, as claimed. This completes the proof of the first part of the theorem.

4. EXAMPLES

We shall now give several examples in which A(RG) contains no nonzero nilpotent elements. We have already remarked in Section 1 that this is the case whenever the order of G is a unit in R.

We obtain a less trivial example by choosing G cyclic of order p, and taking $R = Z_p$. Here the Krull-Schmidt theorem is valid for RG-modules. Furthermore (see [7]), the mapping that assigns to each R-free RG-module M the $\overline{R}$G-module M/pM induces a monomorphism of A(RG) into A($\overline{R}$G). Since Green has shown that A($\overline{R}$G) contains no nonzero nilpotent elements, the same is true for A(RG).

For another example, take G cyclic of order pn', where (n', p) = 1, and let $R = Z_p^*$. Since R is a complete discrete valuation ring, again the Krull-Schmidt theorem holds in this case. Let G be a direct product $G_1 \times G_2$ of a cyclic group G_1 of order p and a cyclic group G_2 of order n'. From [1] or [6] it follows that each indecomposable R-free RG-module M is uniquely expressible in the form $M_1 \otimes_R M_2$, where M_1 is an indecomposable RG_1-module and M_2 is an irreducible RG_2-module, and where

$$(g_1, g_2)\cdot(m_1 \otimes m_2) = g_1 m_1 \otimes g_2 m_2 \qquad (g_i \in G_i,\ m_i \in M_i,\ i = 1, 2).$$

This shows that A(RG) is the direct product of the rings $A(RG_1)$ and $A(RG_2)$. Since neither of these rings contains a nonzero nilpotent element, A(RG) has the same property.

5. THE CYCLIC GROUP OF ORDER FOUR

Let G be a cyclic group of order 4, with generator g; let $R = Z_2$, $\bar{R} = R/2R$, $\Gamma = RG$, $\bar{\Gamma} = \bar{R}G$. The Krull-Schmidt theorem holds for Γ-modules, and we shall now compute the multiplication table of A(Γ) by using equation (6).

Up to isomorphism, there are precisely 9 indecomposable R-free Γ-modules, as has been shown by Troy [12] and Roĭter [9]. We list these modules as

$$R,\ S,\ T,\ (R, S),\ (R, T),\ (S, T),\ X,\ Y,\ \Gamma.$$

Here, R is the trivial Γ-module, while S and T are given by

$$S = R\beta,\quad g\beta = -\beta;\quad T = Rt \oplus Ru,\quad gt = u,\quad gu = -t.$$

The module (R, S) is the unique nonsplit extension of the submodule R by the factor module S. The modules (R, T) and (S, T) are defined analogously. We have already defined the modules X and Y in Section 2.

For convenience, we write ⊗ instead of $\otimes_R$ or $\otimes_{\bar{R}}$, since it will be clear from the context which is intended. If M is any R-free Γ-module, and m = (M: R), there are the obvious relations

$$(10)\qquad R \otimes M \cong M,\qquad \Gamma \otimes M \cong \Gamma^{(m)}\ \ (\text{direct sum of } m \text{ copies of } \Gamma).$$

Straightforward simple calculations yield

$$(11)\qquad \begin{cases} S\otimes S \cong R,\quad S\otimes T\cong T,\quad S\otimes(R,S)\cong(R,S),\quad S\otimes(R,T)=(S,T),\\ S\otimes(S,T)=(R,T),\quad T\otimes T\cong(R,S)^{(2)},\quad T\otimes(R,S)=T^{(2)},\\ (R,S)\otimes(R,S)\cong(R,S)^{(2)}. \end{cases}$$

Denote the elements {R}, {S}, ⋯, {Γ} of A(Γ) by $c_1, c_2, \cdots, c_9$, for convenience. We shall verify the following multiplication table for A(Γ):

(12)

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
c_2	c_1							
c_3	c_3	$2c_4$						
c_4	c_4	$2c_3$	$2c_4$					
c_5	c_6	$c_4 + c_9$	$c_3 + c_9$	$c_1 + 2c_9$				
c_6	c_5	$c_4 + c_9$	$c_3 + c_9$	$c_2 + 2c_9$	$c_1 + 2c_9$			
c_7	c_7	$c_3 + c_4 + c_9$	$c_3 + c_4 + c_9$	$c_8 + 2c_9$	$c_8 + 2c_9$	$c_7 + c_8 + 2c_9$		
c_8	c_8	$c_3 + c_4 + c_9$	$c_3 + c_4 + c_9$	$c_7 + 2c_9$	$c_7 + 2c_9$	$c_7 + c_8 + 2c_9$	$c_7 + c_8 + 2c_9$	
c_9	c_9	$2c_9$	$2c_9$	$3c_9$	$3c_9$	$4c_9$	$4c_9$	$4c_9$

We already know that

$$c_i c_j = c_j c_i, \quad c_i c_1 = c_i \quad (1 \leq i, j \leq 9),$$

and the products $c_i c_9$ are obtained immediately from the second relation in (10). Relations (11) give the products $c_2 c_j$ $(2 \leq j \leq 6)$, c_3^2, $c_3 c_4$, c_4^2.

We shall use equation (6) to compute the products that involve c_7 and c_8. We need some elementary facts about $\overline{\Gamma}$-modules. Set

$$A_j = \overline{\Gamma}/(g-1)^j \overline{\Gamma} \quad (1 \leq j \leq 4).$$

Then A_j is an indecomposable $\overline{\Gamma}$-module of $\overline{R}$-dimension j, and $\{A_j\colon 1 \leq j \leq 4\}$ is a full set of indecomposable $\overline{\Gamma}$-modules. We see that $A_4 = \overline{\Gamma}$, while A_1 is the trivial $\overline{\Gamma}$-module.

Either by an easy direct calculation, or else by referring to Green [4], we obtain the isomorphisms

$$(13) \quad A_1 \otimes A_j \cong A_j, \quad A_4 \otimes A_j \cong A_4^{(j)}, \quad A_2 \otimes A_3 \cong A_2 \oplus A_4, \quad A_3 \otimes A_3 \cong A_1 \oplus A_4^{(2)}.$$

Furthermore, we see that

$$(14) \qquad \overline{R} \cong \overline{S} \cong A_1, \quad \overline{(R, S)} \cong \overline{T} \cong A_2, \quad \overline{(R, T)} \cong \overline{(S, T)} \cong A_3.$$

The discussion in Section 2 shows that

$$(15) \qquad \overline{X} \cong \overline{Y} \cong A_1 \oplus A_3.$$

Now X is the ideal of Γ generated by 2 and g - 1, and Y is the ideal generated by 2 and $1 + g + g^2 + g^3$. Let W be the ideal generated by 2 and $(g-1)^2$. Then X and Y are indecomposable, whereas

$$W \cong (R, S) \oplus T.$$

We may write four exact sequences

$$(16) \qquad \begin{cases} 0 \to X \to \Gamma \to A_1 \to 0, & 0 \to W \to \Gamma \to A_2 \to 0, \\ 0 \to Y \to \Gamma \to A_3 \to 0, & 0 \to \Gamma \to \Gamma \to A_4 \to 0, \end{cases}$$

where the embedding $\Gamma \to \Gamma$ in the last sequence is given by $\gamma \to 2\gamma$ $(\gamma \in \Gamma)$.

Now let M be any R-free Γ-module of R-rank m, and let

$$\overline{M} \otimes A_1 \cong \overline{M} \cong \sum_{i=1}^{4}{}^{\oplus} A_i^{(r_i)}.$$

Equation (6) then yields the relation

$$(17) \qquad M \otimes X \oplus \Gamma^{(r_1+r_2+r_3+r_4)} \cong X^{(r_1)} \oplus W^{(r_2)} \oplus Y^{(r_3)} \oplus \Gamma^{(r_4)} \oplus \Gamma^{(m)}.$$

Let us compute $S \otimes X$; since $\overline{S} \cong A_1$, we get $r_1 = 1$, $r_2 = r_3 = r_4 = 0$, $m = 1$, and so

$$S \otimes X \oplus \Gamma \cong X \oplus \Gamma.$$

This shows that $c_2 c_7 = c_7$. Likewise, $\overline{T} \cong A_2$ gives

$$T \otimes X \oplus \Gamma \cong W \oplus \Gamma^{(2)},$$

so that $c_3 c_7 = c_3 + c_4 + c_9$. As a last illustration, the isomorphism $\overline{X} \cong A_1 \oplus A_3$ yields

$$X \otimes X \oplus \Gamma^{(2)} \cong X \oplus Y \oplus \Gamma^{(4)},$$

whence $c_7^2 = c_7 + c_8 + 2c_9$. In this manner, we evaluate all products involving c_7.

In order to compute $M \otimes Y$, we set

$$\overline{M} \otimes A_3 \cong \sum_{i=1}^{4}{}^{\oplus} A_i^{(s_i)}.$$

Equation (6) now becomes

$$M \otimes Y \oplus \Gamma^{(s_1+s_2+s_3+s_4)} \cong X^{(s_1)} \oplus W^{(s_2)} \oplus Y^{(s_3)} \oplus \Gamma^{(s_4)} \oplus \Gamma^{(m)}.$$

Thus, $\overline{S} \cong A_1$ implies that $\overline{S} \otimes A_3 \cong A_3$, and so

$$S \otimes Y \oplus \Gamma \cong Y \oplus \Gamma,$$

that is, $c_2 c_8 = c_8$. Similarly,

$$\overline{T} \otimes A_3 \cong A_2 \otimes A_3 \cong A_2 \oplus A_4,$$

and this implies that

$$T \otimes Y \oplus \Gamma^{(2)} \cong W \oplus \Gamma \oplus \Gamma^{(2)}.$$

Therefore $c_3 c_8 = c_3 + c_4 + c_9$. Continuing in this manner, we easily evaluate all products involving c_8.

We are left with the problem of computing products such as $(S, T) \otimes (R, T)$. Direct calculation of these is a rather tedious process, and a better approach is to use the ideas of Section 2. There exist exact sequences

(18) $$0 \to (R, T) \to \Gamma \to S \to 0,$$

(19) $$0 \to R \to \Gamma \to (S, T) \to 0,$$

(20) $$0 \to (R, S) \to \Gamma \to T \to 0,$$

(we omit the proof of this.) Tensoring the first of these sequences with (R, T), and using the isomorphism $(R, T) \otimes S \cong (S, T)$, we obtain an exact sequence

$$0 \to (R, T) \otimes (R, T) \to \Gamma^{(3)} \to (S, T) \to 0.$$

The application of Schanuel's lemma to this last sequence and to the sequence in (19) shows that

$$(R, T) \otimes (R, T) \cong R \oplus \Gamma^{(2)}.$$

Thus $c_5^2 = c_1 + 2c_9$. Since $c_6 = c_2 c_5$, we deduce that

$$c_6^2 = c_5^2, \qquad c_5 c_6 = c_2 \cdot c_5^2 = c_2 (c_1 + 2c_9) = c_2 + 2c_9.$$

Similarly, tensoring (18) with T, we obtain an exact sequence

$$0 \to (R, T) \otimes T \to \Gamma^{(2)} \to T \to 0.$$

Comparing this with sequence (20), and using Schanuel's lemma once more, we have the isomorphism

$$(R, T) \otimes T \cong (R, S) \oplus \Gamma.$$

Therefore $c_3 c_5 = c_4 + c_9$. Multiply this last equation by c_2, thereby getting $c_3 c_6 = c_4 + c_9$.

Finally, we may compute $c_4 c_5$ as follows:

$$2c_4 c_5 = c_3^2 c_5 = c_3 (c_4 + c_9) = 2c_3 + 2c_9,$$

whence $c_4 c_5 = c_3 + c_9$. We multiply this last equation by c_2, getting

$$c_4 c_6 = c_3 + c_9.$$

This completes our verification of table (12).

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